EIGENVALUES OF RELAXED COMPACT TORI OF ARBITRARY CROSS-SECTION

Sherif M. Khalil
Plasma & Nuclear Fusion Dept., N.R.C., Atomic Energy Authority, Cairo – Egypt

1. Introduction. It is well known that the equations

$$\vec{J} = \vec{\nabla} x \vec{B} = \mu \vec{B}, \quad \vec{\nabla}\mu = 0 \tag{1}$$

describe the relaxation of toroidal plasmas to a force-free configuration of minimum energy [1,2]. Solution of equations (1) is important for describing the gross features of reversed field pinches (RFP), spheromak configurations, current limitation in toroidal plasmas and others. Two parameters determine the relaxed state of a toroidal system with a perfectly conducting boundary. Firstly, is the magnetic helicity (gauge invariant) $K = \int \vec{A} \cdot \vec{B} \; d\tau$, and secondly is the toroidal flux ψ. Also, it is very important to know the parameter μ (the eigenvalue corresponding to a relaxed plasma state). In relaxed states μ cannot exceed the smallest eigenvalue $\mu_{\min}$, and for a toroidal discharge there is a maximum toroidal current $I_{\max}$, which is connected to μ. The eigenvalue μ and the normalized field profile are determined by the dimensionless ratio K/ψ^2 [3]. The value of μ varies continuously with this ratio but is always below the lowest eigenvalue $\mu_{\min}$ in relaxed state equation (1). When the toroidal flux ψ vanishes, and we have a perfect conducting wall, $\mu_{\min}$ will satisfy the boundary condition:

$$\vec{B} \cdot \vec{n} = 0 \tag{2}$$

To select the correct minimum energy solution of (1) and (2), one must consider the eigenvalues of this system [4]. Solutions of (1) and (2) seem to be difficult in toroidal coordinates. Therefore, the force-free magnetic field boundary value problem has been solved in toroidal coordinates using approximate analytical methods, considering a large, but finite, aspect ratio [5,6]. Toroidal force-free equilibrium has also been considered precisely for tori of rectangular cross-section and finite aspect ratio, and for more general boundaries in the infinite aspect ratio limit [4,6]. Eigenvalues associated with relaxed force-free toroidal plasmas were calculated for tori of arbitrary aspect ratio and arbitrary cross section (e.g., circle, ellipse, rectangle…) [7,8]. The results obtained in this case are in agreement with those obtained previously in the limit when the aspect ratio has a very large value [9].

2. Relaxed state solutions for axisymmetric CT. To find the solution of equation (1) for axisymmetric plasmas, we follow the same methods as per reference [7], and use cylindrical coordinates r, φ, z (Fig.1). In a toroidal container, and for finite aspect ratio, the periodicity condition is expressed as $\vec{B}(r,\varphi,z) = \vec{B}(r,\varphi+2\pi,z)$. Assume a separation of variables as $F(r,\varphi,z) = f(r)\exp i(m\varphi + kz)$. The meridional cross-section in the r, z plane of the toroidal metallic vessel wall shall be described by an arbitrary curve $z = z(r)$ along which the boundary condition (2) has to be satisfied. Along such an arbitrary curve, and taking into consideration that the field lines are tangential to this curve, equation (2) takes the form:

$$B_r(r,z)\frac{dz}{dr} = B_z(r,z) \tag{3}$$

Later, the curve $z = z(r)$ will be represented by a set of points (collocation points) r_i, z_i that lay on the cross-section boundary. From (1), the following expressions linking the magnetic field components are obtained:

$$i\frac{m}{r}B_z - ikB\varphi = \mu B_r \ , \quad ikB_r - \frac{\partial B_z}{\partial r} = \mu B\varphi \, , \quad \frac{1}{r}\frac{\partial (rB_\varphi)}{\partial r} - i\frac{m}{r}B_r = \mu B_z \qquad (4)$$

For an axisymmetric torus, inserting the first two equations from (4) into (3), we obtain [7]:

$$\frac{\partial (rB_\varphi)}{\partial z}dz + \frac{\partial (rB_\varphi)}{\partial r}dr = d(rB_\varphi) = 0 \, , \ \left(\frac{\partial}{\partial \varphi} \to 0, \quad m = 0 \right) \ \text{or} \quad rB_\varphi = Const. = C \qquad (5)$$

Relation (5) coincides with the zero field condition (2) for $C = 0$. The solution of equations (4) gives the magnetic field components for an axisymmetric toroidal plasma [7]

$$B_r(r,z) = \sum_{l=1}^{N} k_l Sin(k_l z)\left[a_l J_1(K_l r) + b_l Y_1(K_l r)\right] \qquad (6)$$

$$B_z(r,z) = \sum_{l=1}^{N} \mathrm{K}_l Cos(k_l z)\left[a_l J_0(\mathrm{K}_l r) + b_l Y_0(\mathrm{K}_l r)\right] \qquad (7)$$

$$B_\varphi(r,z) = \sum_{l=1}^{N} \mu Cos(k_l z)\left[a_l J_1(\mathrm{K}_l r) + b_l Y_1(\mathrm{K}_l r)\right] \qquad (8)$$

where J_0, J_1 are Bessel functions; Y_0, Y_1 are Neumann functions; a_l, b_l are constant coefficients; and $\mathrm{K}_l = \sqrt{\mu^2 - k_l^2}$. It should be mentioned here that, for the case of a straight cylinder (or for infinite aspect ratio of toroidal plasma), the coefficients b_l go to zero, since the Neumann functions are singular at $r = 0$. On the other hand, for finite aspect ratio or CT, as in our case, one has to keep these functions. Inserting (8) into (5), the boundary condition for an axisymmetric container of finite aspect ratio and arbitrary cross-section reads:

$$\sum_{l=1}^{N} r\mu \quad Cos(k_l z)\left[a_l J_1(\mathrm{K}_l r) + b_l Y_1(\mathrm{K}_l r)\right] \ = C \qquad (9)$$

Equation (9) describes the magnetic field lines (along which the normal component of the magnetic field vanishes). On a perfectly conducting wall the normal component of $\vec{B}$ vanishes too. For $C = 0$ equation (9) will coincide with the zero field condition.

3. Numerical method. The boundary condition (9) represents N homogeneous equations, each having two partial coefficients a_l, b_l (i.e., altogether 2N coefficients). These equations should be satisfied for continuously varying r, z along any arbitrary curve $z = z(r)$. The method used is to consider a finite number of points *P* (collocation points r_i, z_i) for which it is assumed that the boundary condition (9) is satisfied. The accuracy of the result apparently depends on *P*. For fixed *P*, it is assumed that the accuracy is also constant. Instead of solving the transcendental system (9), according with experience with numerical approximation [8,9], it is easier to choose arbitrary but evenly distributed values of wave numbers k_l between zero and a guess for eigenvalues μ (in order to ensure real values of k_l). Later, we shall show that for arbitrary different values of k_l in the range $0 - \mu$, the eigenvalues are not changing.

We have now 2*N* +1 unknowns (namely a_l, b_l and μ) with 2*N* linear homogeneous algebraic equations, the vanishing of their determinant yielding the lowest eigenvalue. Then, we may choose one coefficient with known value, e.g., $a_l = 1$, and one equation is omitted. Accordingly, the remaining 2*N*-1 inhomogeneous algebraic equations can be solved numerically for the 2*N*-1 unknowns, i.e., $(a_2, a_2, ..., a_N, b_1, b_2, ...b_N)$. The fixed number *P* of collocation points determines the number *N* of partial solutions through *N* = *P*/2. This method has been successfully applied to tori of arbitrary curved meridional cross sections and arbitrary aspect ratio [7].

4. Eigenvalues for CT with arbitrary cross-section. Let us consider a CT with circular cross-section. In this case, z in (9) can be represented by:

$$z_i(r) = \sqrt{a^2 - (r_i - R)^2}, \quad i = 1,2,........P \tag{10}$$

Since rB_φ is symmetric in z but not in r, the collocation points have been distributed over the whole upper half of the cross-section. For infinite aspect ratio, the boundary condition reads: $B_\varphi = a_1 J_0(\mu a) = 0$, which yields eigenvalue $\mu a = 2.4048$ [4]. Table 1 gives the lowest eigenvalues, for CT of circular cross-section and aspect ratios $\alpha = 2,1.5$ ($R = 2,1.5$ and $a = 1$). Increasing the aspect ratio to $\alpha = 10,20$ (cylindrical approximation), the zero field eigenvalue μa decreases and tends to the value 2.4048 [7] for axisymmetric $m = 0$ mode, which was obtained by Taylor [2,4] for infinite aspect ratio. For circular cross-section μ increases with the decrease of α. One might argue that the choice of wave numbers k_l is highly arbitrary and influences the result for μ. Hovewer, for several runs of the program with various k_l it is found that μ is insensitive to the choice of k_l as indicated in Table 1.

k_1	k_2	k_3	k_4	k_5	k_6	$\alpha = 2$	$\alpha = 1.5$	**Table 1**
0.001	0.4	0.6	1.7	1.9	2.0	μ=2.44692724	μ=2.47372813	
0.01	0.1	1.0	1.5	2.0	2.1	μ=2.44692709	μ=2.47372830	
0.1	0.5	0.6	1.3	1.8	2.0	μ=2.44692762	μ=2.47372850	
1.0	1.3	1.5	1.7	1.9	2.1	μ=2.44692696	μ=2.47372901	

So far, all the results obtained have been calculated for a circle represented by the collocation points derived from relation (10). This circle, along which the boundary condition is fulfilled, must also be described by (10) for all points laying on it. Therefore, we plotted the function given by equation (9) for $C = 0; C \neq 0$. The result is shown in Fig. 2 for $R = 1.5, a = 1$, $N = 6$ modes ($k_l = 0.001,0.4,0.6,1.7,1.9,2.0$) and eigenvalue $\mu a = 2.47372813$. The collocation points r_i, z_i are marked by small squares and it is clear that they are well fitted on the boundary. For $C = 0$, this gives the poloidal magnetic field line along the container, and when C has different values, this gives the magnetic field lines inside the container (numbers written on the field lines are the different values of C). The outward toroidal shift is clearly seen for small aspect ratio. The above procedure is also performed for other cross-sections. A multipinch is described here by a Cassini curve shifted by the torus major radius R: $(u^2 + z^2)^2 + 2b^2 (u^2 - z^2)^2 - a^4 + b^4 = 0$, $r - R = u$. The Cassini curve is identical to a multipinch configuration if $b < a < b\sqrt{2}$ with half-width given by $\Delta^2 = a^2 - b^2$. Figure 3 shows a Cassini curve container with $\mu = 3.2133226$, $R = 1$, $a = 0.35$, $b = 0.3$, $N = 6$ modes ($k_l = 0.01,0.1,1.0,1.3,1.8,2.1$). When $R = 2$, $a = 1$, $b = 0.5$ the corresponding eigenvalue becomes $\mu = 2.50059$ [7]. A D-shaped cross-section is also investigated in Fig. 4, with an eigenvalue $\mu = 1.5050883$. A trial is also made to apply the above methods to a sharp edged cross section (e.g., astroidal-like shape). Results are given in Table 2, and Fig. 5. It is clear that the magnetic field lines are smoothly connected at the vortices.

k_1	k_2	k_3	k_4	k_5	k_6	$R = 2; a = b = 1.5$	$R = 2; a = 1; b = 1.5$	**Table 2**
0.001	0.3	0.5	1.6	1.7	1.8	μ=3.21633644	μ=2.66830312	
0.01	0.1	1.1	1.4	1.6	1.7	μ=3.21633240	μ=2.66831130	
0.1	0.5	0.6	1.3	1.5	1.7	μ=3.21633541	μ=2.66830214	
1.0	1.3	1.5	1.6	1.7	1.9	μ=3.21633512	μ=2.66830453	

5. Conclusion. In conclusion, it is shown that the numerical method (collocation method) [7,10] works quite well for CT with tight aspect ratio and arbitrary cross section. It gives, with high accuracy, the zero field eigenvalues of the relaxed force-free equation. A good fulfillment of the boundary condition, which describes the relaxed state along the whole boundary for different cross-sections, is achieved. It would be also of interest to check our methods for the cases when $\alpha \to 1$, coupling magnetic and fluid aspects of plasma (i.e., for $\vec{J}x\vec{B} \cong \vec{\nabla}P \neq 0$) and for nonaxisymmetric cases, which are under investigation.

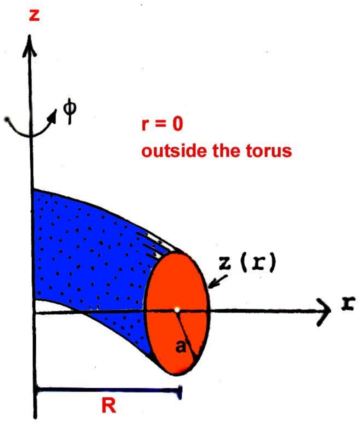
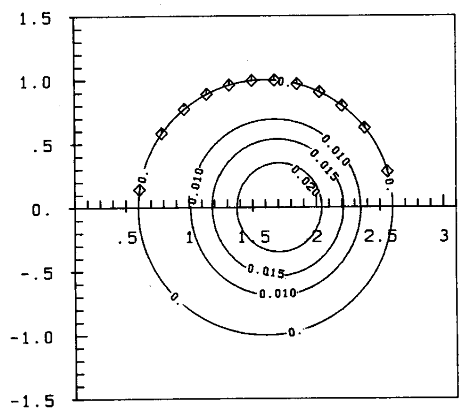
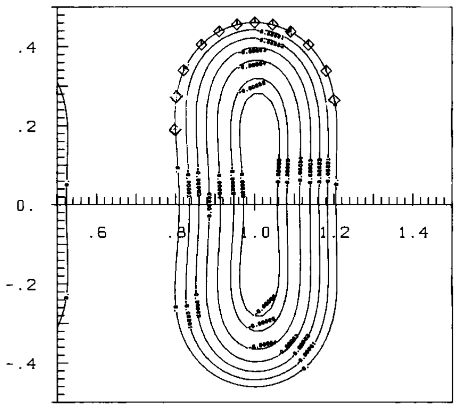


Fig. 1



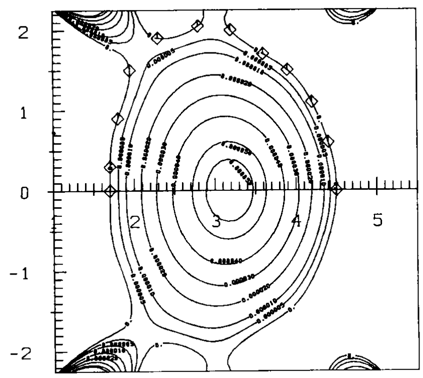
CT of Circular Cross-Section

Fig. 2



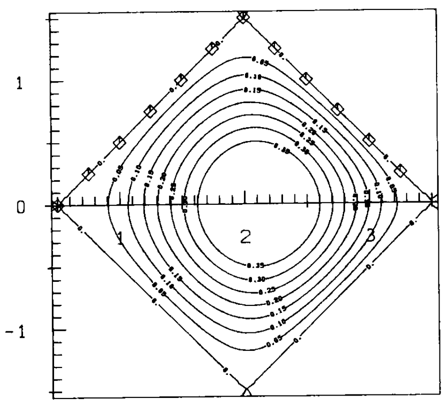
CT of Cassini Cross-Section (Multipinch)

Fig. 3



CT of D-Shaped Cross Section

Fig. 4



CT of Astroida-Like Cross-Section

Fig. 5

References:

[1] J.B. Taylor ; *Plasma Physics and Controlled Nuclear Fusion Research* , 5th Conference Proceedings (IAEA), Tokyo, 1, 388 (1975).

[2] J. B. Taylor, *Rev. Mod. Phys.* 58, 741 (1986).

[3] J. B. Taylor and M.F. Turner ; *Nuclear Fusion* , 29 , 1166 (1989).

[4] C.G. Gimblett, P.J. Hall, J.B. Taylor and M.F. Turner ; *Phys. Fluids*, 30,3186 (1987).

[5] G. Buck, *J. Appl. Phys.* 36, 2231 (1965).

[6] L. Turner, *Phys. Fluids* 27, 1677 (1984).

[7] F. Cap and Sh.M. Khalil, *Nuclear Fusion* ; 29, 1166 (1989).

[8] F. Cap, *Comput. Phys. Commun.* 40, 99 (1986).

[9] J. B. Taylor, *Phys. Rev. Lett.* 33, 1139 (1974).

[10] F. Cap, *4th Workshop on Plasma and Laser Physics*, Cairo - Egypt, Feb. 26-29 (1996), International

ACKNOWLEDGMENT: The author would like to thank the IAEA, Physics Section, for the assistance and support which enabled him to attend and participate in the "2nd IAEA TCM on ST and 7th STW". Also, thanks for the kind hospitality of the organizers (INPE).