Chandrasekhar-Kendall-Furth Configurations for Magnetic Confinement

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1. Chandrasekhar-Kendall-Furth (CKF) Force-Free Fields

A simply connected magnetic confinement scheme can be obtained superposing two axisymmetric homogeneous force-free fields, each with $\vec{\nabla} \wedge \vec{B} = \mu \vec{B}$, both having the same value of the relaxation parameter $\mu = \mu_0 \vec{j} \cdot \vec{B} / B^2$. The first is the Chandrasekhar-Kendall force-free field [1] of order-1, which in spherical geometry (r, ϑ, ϕ_G) admits the poloidal flux function: $\psi^{CK}_{\mu,1}(r,\vartheta) = -(\mu r)\, j_1(\mu r) \sin\vartheta\, P^1_1(\cos\vartheta)$, where $j_1(\mu r)$ is the spherical Bessel function of order 1, having its m-th radial zero at $(\mu r) = x_{1,m}$ and $P^1_1(\cos\vartheta)$ is the Legendre polynomial. The second is the Furth square-toroid force-free field [2], which can be written as: $\psi^F_{\mu,\lambda}(r,\vartheta) = \sqrt{\mu^2 - \lambda^2}\, r \sin\vartheta\, J_1\left(\sqrt{\mu^2 - \lambda^2}\, r \sin\vartheta\right) \cos(\lambda r \cos\vartheta)$, for any value of $\lambda < \mu$, where J_1 is the cylindrical Bessel function. The superposition of the two force-free fields is written: $\psi(r,\vartheta) = \psi^{CK}_{\mu,1} + \gamma\, \psi^F_{\mu,\lambda}$ (Fig. 1). For values of the superposition constant $\gamma \geq 0.402\ldots$, it contains, in a simply connected region near the origin, a toroidal current density j_ϕ of the same sign and can be called a Chandrasekhar-Kendall-Furth force-free field (CKF).

Figure 2 shows the cross-section of the CKF force-free field with superposition constant $\gamma = 0.55$ and details its composition in terms of different plasma regions, divided by a magnetic separatrix. The "main spherical torus" (ST) has a safety factor (inverse of the rotational transform, $q = 1/\iota$), which is $q^{ST}_0 \sim 1.0$ on the magnetic axis and $q^{ST}_{95} \sim 1.5$ at the edge (95% of the poloidal flux of the magnetic separatrix). The two "secondary tori" (SC), present on top and bottom of the main torus, also have $q^{SC}_0 \sim 1.0$ on their magnetic axes and $q^{SC}_{95} \sim 1.5$ at their edges. The discharge surrounding the three tori, which will be dubbed as "spheromak" (P) has a larger safety factor, respectively $q^P_0 \sim 1.5$ on the symmetry axis and $q^P_{95} \sim 3.7$ at the separatrix. When the superposition constant exceeds $\gamma = 0.69\ldots$, the secondary tori disappear (see Fig. 1).

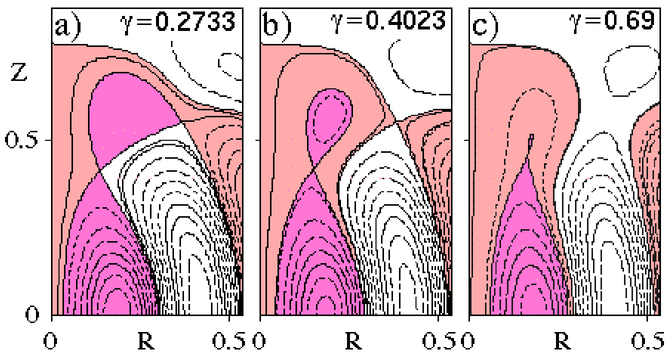

Fig. 1. Poloidal flux function contours of CKF force-free fields $\psi = \psi^{CK}_{\mu,1} + \gamma \psi^F_{\mu,\lambda}$, The superposition constants are: (a) $\gamma = 0.273$; (b) $\gamma = 0.402$; (c) $\gamma = 0.690$. Colored areas mean toroidal current density $j_\phi > 0$; white areas mean $j_\phi < 0$.

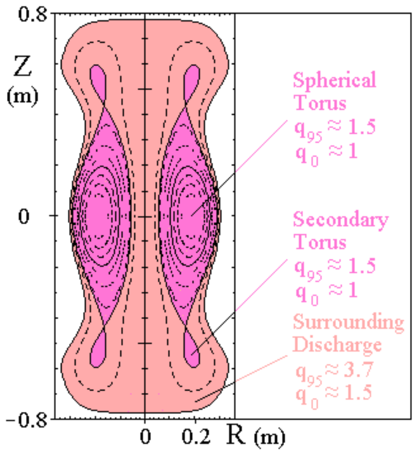

Fig. 2. Contours of the poloidal flux function of the Chandrasekhar-Kendall-Furth force-free field $\psi = \psi^{CK}_{\mu,1} + \gamma \psi^F_{\mu,\lambda}$, with superposition constant $\gamma = 0.55$.

2. Ideal MHD Stability of CKF Force-Free Fields

The ideal MHD stability of the Chandrasekhar-Kendall-Furth (CKF) force-free fields has been studied by solving the eigenvalue problem: $\overleftrightarrow{\mathbf{W}} \cdot \vec{\xi} = \omega^2 \overleftrightarrow{\mathbf{K}} \cdot \vec{\xi}$, where $\overleftrightarrow{\mathbf{W}}$ is the plasma perturbed potential energy and $\overleftrightarrow{\mathbf{K}}$ the plasma perturbed kinetic energy, associated with the perturbed plasma displacement $\vec{\xi}$, and ω^2 is the eigenvalue. The expressions for the perturbed energies become simpler [3] if the equilibrium is analyzed in non-orthogonal periodical Boozer coordinates (ψ_T-radial, θ-poloidal, ϕ-toroidal) [4], with Jacobian $\sqrt{g} \propto 1/B^2$. The radial variable ψ_T is the toroidal flux divided by 2π, with $\psi_T=0$ on the magnetic axis of the main torus, $\psi_T=\psi_T^X$ on the separatrix and $\psi_T=\psi_T^{max}$ at the edge of the surrounding spheromak and on the symmetry axis. The continuity of the rotational transform $\iota(\psi_T)$ and of the toroidal and poloidal plasma currents $I(\psi_T)$ and $f(\psi_T)$ joins the Boozer coordinates of these equilibria at the ST-P interface, $\psi_T=\psi_T^X$ (see Fig. 3).

The energy principle for a compressible plasma [5] is used, Fourier analyzing the normal $\xi^\psi=\vec{\xi}\cdot\vec{\nabla}\psi_T$, binormal $\eta=\vec{\xi}\cdot(\vec{\nabla}\theta - \iota\vec{\nabla}\phi)$ and parallel $\mu=-\sqrt{g}\,\vec{\xi}\cdot\vec{\nabla}\phi$ components of the displacement away from the (up/down symmetric) equilibrium as: $\xi^\psi = \Sigma_l\, \xi_l(\psi_T)\sin(m_l\theta - n\phi)$, $\eta = \Sigma_l\, \eta_l(\psi_T)\cos(m_l\theta - n\phi)$, $\mu = \Sigma_l\, \mu_l(\psi_T)\cos(m_l\theta - n\phi)$, in terms of a toroidal mode number n and of a superposition of poloidal mode numbers m_l. Only the normal displacement ξ^ψ must be continuous [5] at the separatrix between the three tori and the surrounding spheromak, whereas the binormal η and the tangential μ components can make jumps at $\psi_T=\psi_T^X$.

The poloidal angle θ makes an excursion $[0, \theta_X]$ on the lower outboard of the main spherical torus and then runs in the range $[\theta_X, 2\pi+\theta_X]$ in the lower secondary torus. It continues to the angles $[2\pi+\theta_X, 4\pi-\theta_X]$ on the inboard of the main spherical torus, runs in the range $[4\pi-\theta_X, 6\pi-\theta_X]$ in the upper secondary torus and finishes its run in the range $[6\pi-\theta_X, 6\pi]$ on the upper outboard of the main spherical torus. Inside the surrounding spheromak it simply covers the range $[0, 2\pi]$. So a correspondence is established between the poloidal angle θ inside the tori and the poloidal angle $3\cdot\theta$ inside the surrounding spheromak. MHD "global" modes, which exist over the whole plasma, must have periodical perturbed displacements in terms of the Boozer poloidal angle θ. For these global modes the allowed poloidal mode numbers can be all the integers $m_l=\ldots,-2,-1,0,1,2,3\ldots$ inside the main and the secondary tori.

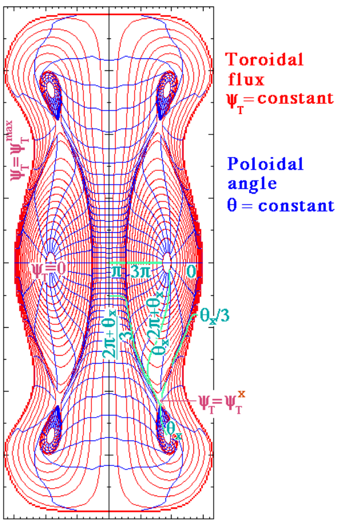

Fig. 3. Boozer coordinates of CKF force-free field with superposition constant γ=0.55.

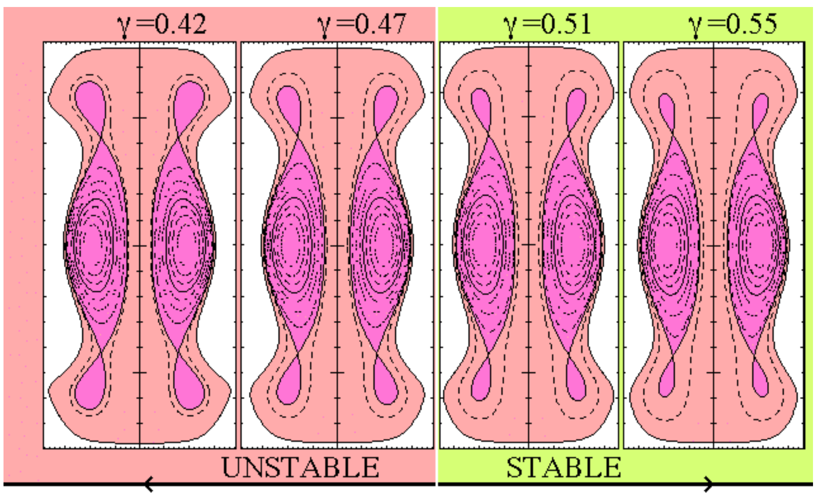

Fig. 4. Sequence of CKF force-free fields with ideal MHD stability boundary as a function of the superposition parameter γ.

Therefore the continuity of ξ^{ψ} forces the allowed poloidal mode numbers of the global perturbations to be limited to multiples of 3: $m_l=\ldots,-6,-3,0,3,6,9,\ldots$ inside the surrounding spheromak. However other MHD "internal" modes can still exist if their radial extent is limited to the surrounding spheromak, $\psi_T^X \le \psi_T \le \psi_T^{max}$. These internal perturbations have as allowed poloidal mode numbers all the integers $m_l=\ldots,-2,-1,0,1,2,3,\ldots$, also inside the surrounding spheromak: their radial ξ^{ψ} component will go to zero at the magnetic separatrix, $\xi^{\psi}(\psi_T^X)=0$, and will not have to match any requirement of periodicity on the three tori.
The result of the ideal MHD stability calculations for low toroidal mode numbers (n=1,2,3), assuming fixed boundary conditions at the edge of the plasma: $\xi^{\psi}(\psi_T^{max})=0$, is that the Chandrasekhar-Kendall-Furth force-free fields are stable when the value of the superposition parameter is greater than $\gamma=0.5$ (see Fig. 4).

3. Unrelaxed Finite β Configurations Similar to CKF Force-Free Fields

However force-free fields have $\vec{\nabla}p=0$ and are so unable to confine plasmas of fusion interest. Nevertheless a variety of unrelaxed ($\vec{\nabla}\mu\neq 0$, $\vec{\nabla}p\neq 0$) MHD fixed boundary equilibria, similar in shape and topology to the CKF force-free fields, can be calculated (see Fig. 5).

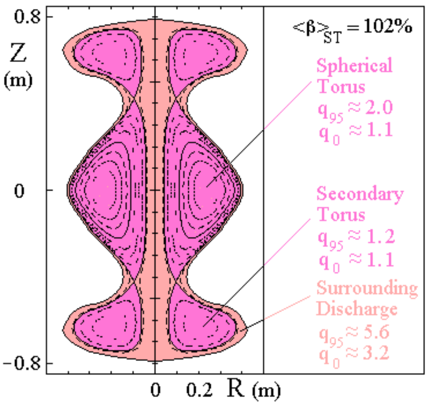

Fig. 5. Flux function contours of an unrelaxed CKF equilibrium, with $<\beta>_{ST}=102\%$.

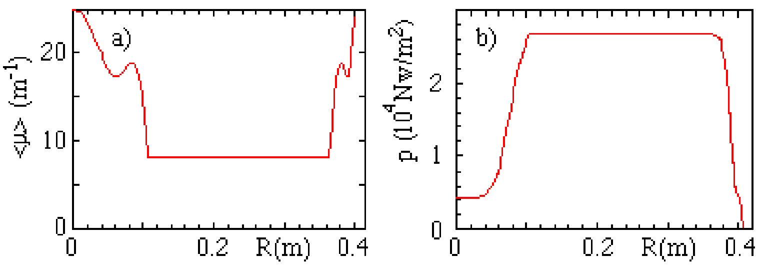

Fig. 6. Profiles of: a) surface averaged $<\mu>$; b) pressure, on the equator of the unrelaxed $<\beta>_{ST}=102\%$ CKF of Fig. 5.

They have $\mu=\mu_0\,\vec{j}\cdot\vec{B}/B^2$=constant only at the edge of the plasma ($\psi_T=\psi_T^{max}$), as a boundary condition for the MHD equilibrium. In the example shown in Fig. 6 the surface averaged value $<\mu>=\mu_0<\vec{j}\cdot\vec{B}/B^2>$ drops from $<\mu>=25$, at the edge of the surrounding spheromak, down to $<\mu>=8$, on the axis of the main spherical torus. Such $<\mu>$ profile corresponds to a sustainment obtained by driving current on the flux surfaces of the surrounding spheromak. Magnetic helicity, flowing down the $<\mu>$ gradient, will be injected into the main spherical torus, through magnetic reconnections at the X-points. The gradient of the pressure profile will presumably be concentrated in the same region where the gradient of $<\mu>$ has the largest variation (see Fig. 6).
The case shown in Figs. 5 and 6 corresponds to a very high beta in the spherical torus $\beta_{ST}=2\mu_0<p>_{ST}/<B^2>_{ST}=102\%$. It refers to an unrelaxed CKF in which the total poloidal spheromak current is I_e=60 kA, whereas the total toroidal current in the main spherical torus is I_p=451 kA. The safety factor of the main spherical torus has the values $q_0^{ST}\sim 1.1$ on the magnetic axis and $q_{95}^{ST}\sim 2.0$ at the edge (95% of the poloidal flux of the magnetic separatrix).

The two secondary tori, present on top and bottom of the main torus, have q_0^{SC}~1.1 on their magnetic axes and q_{95}^{SC}~1.2 at their edges. The spheromak discharge, which surrounds the three tori, has a larger safety factor: respectively q_0^{P}~3.2 on the symmetry axis and q_{95}^{P}~5.6 at the separatrix. An unrelaxed CKF equilibrium can be defined as a spherical torus (q_0^{ST}~1,q_{95}^{ST}~2) enclosed within a spheromak endowed with high elongation and therefore with high winding number (q_0^{P}~3, q_{95}^{P}~5) and intrinsically reversed shear.

The ideal MHD stability of the unrelaxed CKF configurations has been calculated with the same numerical code used for the CKF force-free fields. The ideal MHD stability of unrelaxed CKF equilibria, with profiles of $\langle\mu\rangle(\psi_T)$ and $p(\psi_T)$ similar to the ones shown in Fig. 6, has been calculated with fixed boundary conditions at the edge: $\xi^{\psi}(\psi_T^{max})=0$. Cases, like the one shown in Fig. 5, with a beta of the main spherical torus β_{ST} even exceeding unity, remain stable to all the ideal MHD perturbations with low toroidal mode numbers (n=1,2,3), see Fig. 7. However with different choices of the $\langle\mu(\psi)\rangle$ and $p(\psi)$ profiles (in particular with an excessive $\Delta\langle\mu\rangle$ across the plasma), the beta limit with respect to low-n ideal MHD modes is reduced. A case of an unstable CFK configuration with β_{ST} =67% is shown in Fig. 8.

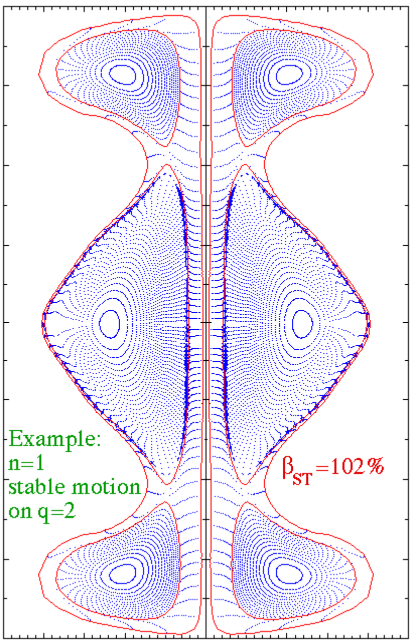

Fig. 7. Fluid displacement plot of a stable CFK.

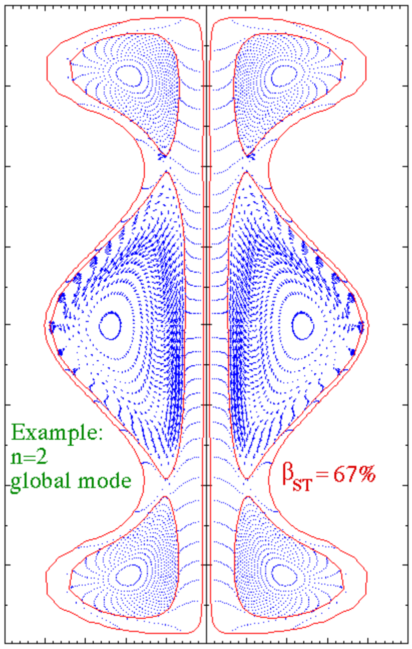

Fig.8. Fluid displacement plot of an unstable CFK.

Reactor extrapolations of unrelaxed CKF magnetic configurations, endowed with the right helicity injection, β limit and energy confinement, will allow for an unimpeded outflow of the high energy charged fusion products. The charged fusion products will drift across the magnetic separatrix, to the degenerate X-points (B=0) on top/bottom of the configuration easing direct energy conversion and the use of the burner as a space thruster.

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