g-gauge 色散关系推导

目录

1、$\gamma(\boldsymbol{x})=\gamma_0+\varepsilon\gamma_1+...$的推导

$$\gamma(\boldsymbol{x})=\gamma(\boldsymbol{X})+\varepsilon i_\rho d\gamma(\boldsymbol{X})+\frac{\varepsilon^2}{2}i_\rho d[i_\rho d\gamma(\boldsymbol{X})]+... \tag{1}$$

$$\gamma(\boldsymbol{X})=(\boldsymbol{A}+\boldsymbol{v})\cdot d\boldsymbol{X}-(\phi+\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2})dt \tag{2}$$

速度表示为以下形式:

$$\boldsymbol{v}=u\boldsymbol{b}+\boldsymbol{w}+\boldsymbol{D}$$

其中定义

$$\boldsymbol{b}(\boldsymbol{X},t)=\frac{\boldsymbol{B}_0}{B_0},$$

$$\boldsymbol{w}(\boldsymbol{X},\mu,\theta,t)=\sqrt{2\mu B_0}\boldsymbol{c},$$

$$\boldsymbol{D}(\boldsymbol{X},t)=\frac{\boldsymbol{E}_0\times\boldsymbol{b}}{B_0},$$

$$\partial_\mu \boldsymbol{w} = \partial_\mu \sqrt{2\mu B_0}\boldsymbol{c} = \sqrt{\frac{B_0}{2\mu}}\boldsymbol{c},$$

$$\partial_\theta \boldsymbol{w} = \partial_\theta \sqrt{2\mu B_0}\boldsymbol{c} = \boldsymbol{w}\times\boldsymbol{b} = \sqrt{2\mu B_0}\partial_\theta \boldsymbol{c} = -\sqrt{2\mu B_0}\boldsymbol{a}$$

$\boldsymbol{\rho} = -\dfrac{\boldsymbol{w}\times\boldsymbol{b}}{B_0}, \boldsymbol{\rho} = \rho\boldsymbol{a} = \sqrt{\dfrac{2\mu}{B_0}}\boldsymbol{a}$ 是定义的在构型空间中由 X->x 的量。只是 $\boldsymbol{\rho} = \rho^X \partial_X$

其中 $\boldsymbol{a},\boldsymbol{c}$ 以及回旋角 $,\theta$ 的定义为：

$$\boldsymbol{a} = \boldsymbol{b}\times\boldsymbol{c} = \boldsymbol{e}_1\cos\theta - \boldsymbol{e}_2\sin\theta$$
$$\partial_\theta \boldsymbol{c} = -\boldsymbol{a},$$
$$\partial_\theta \boldsymbol{a} = \boldsymbol{c}$$

基本的自变量是 $(\boldsymbol{X}, u, \mu, \theta, t)$

将以上定义代入(2)式:

$$\gamma(\boldsymbol{X}) = (\boldsymbol{A} + u\boldsymbol{b} + \boldsymbol{w} + \boldsymbol{D})\cdot d\boldsymbol{X} - (\phi + \frac{(u\boldsymbol{b} + \boldsymbol{w} + \boldsymbol{D})\cdot(u\boldsymbol{b} + \boldsymbol{w} + \boldsymbol{D})}{2})dt$$

$$= (\boldsymbol{A} + u\boldsymbol{b} + \boldsymbol{w} + \boldsymbol{D})\cdot d\boldsymbol{X} - (\phi + \frac{u^2 + w^2 + D^2 + 2\boldsymbol{w}\cdot\boldsymbol{D}}{2})dt$$

$$= (\boldsymbol{A} + u\boldsymbol{b} + \boldsymbol{D})\cdot d\boldsymbol{X} - (\phi + \frac{u^2 + w^2 + D^2}{2})dt + \boldsymbol{w}\cdot d\boldsymbol{X} - \boldsymbol{w}\cdot\boldsymbol{D}dt$$

$$= (\frac{\boldsymbol{A}_0}{\varepsilon} + u\boldsymbol{b} + \boldsymbol{D})\cdot d\boldsymbol{X} - (\frac{\phi_0}{\varepsilon} + \frac{u^2 + w^2 + D^2}{2})dt + \boldsymbol{w}\cdot d\boldsymbol{X} - \boldsymbol{w}\cdot\boldsymbol{D}dt + \ldots$$

令 $\boldsymbol{A}_0^* = \dfrac{\boldsymbol{A}_0}{\varepsilon} + u\boldsymbol{b} + \boldsymbol{D}, \phi_0^* = \dfrac{\phi_0}{\varepsilon} + \dfrac{u^2 + w^2 + D^2}{2}$

（$i_\rho dA = -\boldsymbol{w}\cdot d\boldsymbol{X} + \boldsymbol{w}\cdot\boldsymbol{D}dt$）

$$\gamma(\boldsymbol{X}) = \boldsymbol{A}_0^*\cdot d\boldsymbol{X} - \phi_0^* dt + \boldsymbol{w}\cdot d\boldsymbol{X} - \boldsymbol{w}\cdot\boldsymbol{D}dt + \ldots$$

（$\gamma(\boldsymbol{X}) = \boldsymbol{A} + \boldsymbol{p}_c - \varepsilon i_\rho dA$）

由此得

$$d\gamma(\boldsymbol{X}) = d(\boldsymbol{A}_0^*\cdot d\boldsymbol{X} - \phi_0^* dt + \boldsymbol{w}\cdot d\boldsymbol{X} - \boldsymbol{w}\cdot\boldsymbol{D}dt)$$

前两项：

$$
\begin{aligned}
&d(\boldsymbol{A}_0^* \cdot d\boldsymbol{X}) - d(\phi_0^* dt) \\
&= -\varepsilon_{jk}^i (\nabla \times \boldsymbol{A}_0^*)_i dX^j \wedge dX^k + \partial_t \boldsymbol{A}_0^* dt \wedge d\boldsymbol{X} + \partial_u \boldsymbol{A}_0^* du \wedge d\boldsymbol{X} \\
&-\nabla \phi_0^* d\boldsymbol{X} \wedge dt - \partial_u \phi_0^* du \wedge dt - \partial_\mu \phi_0^* d\mu \wedge dt \\
&= -\varepsilon_{jk}^i \boldsymbol{B}_{0i}^* dX^j \wedge dX^k + (-\partial_t \boldsymbol{A}_0^* - \nabla \phi_0^*) d\boldsymbol{X} \wedge dt \\
&+\boldsymbol{b} du \wedge d\boldsymbol{X} - u du \wedge dt - B_0 d\mu \wedge dt \\
&= -\varepsilon_{jk}^i \boldsymbol{B}_{0i}^* dX^j \wedge dX^k + \boldsymbol{E}_0^* d\boldsymbol{X} \wedge dt \\
&+\boldsymbol{b} du \wedge d\boldsymbol{X} - u du \wedge dt - B_0 d\mu \wedge dt
\end{aligned}
$$

其中 $\boldsymbol{B}_0^* = \nabla \times \boldsymbol{A}_0^*$， $\boldsymbol{E}_0^* = -\partial_t \boldsymbol{A}_0^* - \nabla \phi_0^*$

后两项：

$$
\begin{aligned}
&d(\boldsymbol{w} \cdot d\boldsymbol{X}) - d(\boldsymbol{w} \cdot \boldsymbol{D} dt) \\
&= -\varepsilon_{jk}^i (\nabla \times \boldsymbol{w})_i dX^j \wedge dX^k + \partial_t \boldsymbol{w} dt \wedge d\boldsymbol{X} + \partial_\mu \boldsymbol{w} d\mu \wedge d\boldsymbol{X} + \partial_\theta \boldsymbol{w} d\theta \wedge d\boldsymbol{X} \\
&-\nabla(\boldsymbol{w} \cdot \boldsymbol{D}) d\boldsymbol{X} \wedge dt - \partial_\mu (\boldsymbol{w} \cdot \boldsymbol{D}) d\mu \wedge dt - \partial_\theta (\boldsymbol{w} \cdot \boldsymbol{D}) d\theta \wedge dt \\
&= -\varepsilon_{jk}^i (\nabla \times \boldsymbol{w})_i dX^j \wedge dX^k + \partial_t \boldsymbol{w} dt \wedge d\boldsymbol{X} + \sqrt{\frac{B_0}{2\mu}} \boldsymbol{c} d\mu \wedge d\boldsymbol{X} - \sqrt{2\mu B_0} \boldsymbol{a} d\theta \wedge d\boldsymbol{X} \\
&-\nabla(\boldsymbol{w} \cdot \boldsymbol{D}) d\boldsymbol{X} \wedge dt - (\sqrt{\frac{B_0}{2\mu}} \boldsymbol{c} \cdot \boldsymbol{D}) d\mu \wedge dt + \sqrt{2\mu B_0} \boldsymbol{a} \cdot \boldsymbol{D} d\theta \wedge dt
\end{aligned}
$$

$$\boldsymbol{\rho} = \rho^X \partial_X$$

前两项：

$$
\begin{aligned}
&i_{\boldsymbol{\rho}} d\gamma(\boldsymbol{X})_{1,2} = -\rho^X \times \boldsymbol{B}_0^* \cdot d\boldsymbol{X} + \rho^X \cdot \boldsymbol{E}_0^* dt - \rho^X \cdot \boldsymbol{b} du \\
&= -\rho^X \times (\nabla \times \boldsymbol{A}_0^*) \cdot d\boldsymbol{X} + \rho^X \cdot (-\partial_t \boldsymbol{A}_0^* - \nabla \phi_0^*) dt \\
&= -\rho^X \times [\boldsymbol{B}_0 + \nabla \times (u\boldsymbol{b} + \boldsymbol{D})] \cdot d\boldsymbol{X} + \rho^X \cdot [-\partial_t (\frac{\boldsymbol{A}_0}{\varepsilon} + u\boldsymbol{b} + \boldsymbol{D}) - \nabla(\frac{\phi_0}{\varepsilon} + \frac{u^2 + w^2 + D^2}{2})] dt \\
&= -\rho^X \times \boldsymbol{B}_0 \cdot d\boldsymbol{X} - \rho^X \times [\nabla \times (u\boldsymbol{b} + \boldsymbol{D})] \cdot d\boldsymbol{X} + \rho^X \cdot [\boldsymbol{E}_0 - \partial_t (u\boldsymbol{b} + \boldsymbol{D}) - \nabla(\frac{u^2 + w^2 + D^2}{2})] dt \\
&= -\boldsymbol{w} \cdot d\boldsymbol{X} - \rho^X \times [\nabla \times (u\boldsymbol{b} + \boldsymbol{D})] \cdot d\boldsymbol{X} + \boldsymbol{w} \cdot \boldsymbol{D} dt + \rho^X \cdot [-\partial_t (u\boldsymbol{b} + \boldsymbol{D}) - \nabla(\frac{u^2 + w^2 + D^2}{2})] dt
\end{aligned}
$$

其中

$$\rho^X \times \boldsymbol{B}_0 = \boldsymbol{w}$$

$$\rho^X \cdot \boldsymbol{E}_0 = -\frac{\boldsymbol{w} \times \boldsymbol{b}}{B_0} \cdot \boldsymbol{E}_0 = -\boldsymbol{w} \cdot \frac{\boldsymbol{b} \times \boldsymbol{E}_0}{B_0} = \boldsymbol{w} \cdot \boldsymbol{D}$$

后两项：

$$i_\rho d\gamma(\boldsymbol{X})_{3,4}=-\rho^X\times(\nabla\times\boldsymbol{w})\cdot d\boldsymbol{X}-\rho^X\cdot\partial_t\boldsymbol{w}dt-\rho^X\cdot\sqrt{\frac{B_0}{2\mu}}\boldsymbol{c}d\mu+\rho^X\cdot\sqrt{2\mu B_0}\boldsymbol{a}d\theta-\rho^X\cdot\nabla(\boldsymbol{w}\cdot\boldsymbol{D})dt$$
$$=-\rho^X\times(\nabla\times\boldsymbol{w})\cdot d\boldsymbol{X}-\rho^X\cdot\partial_t\boldsymbol{w}dt+2\mu d\theta-\rho^X\cdot\nabla(\boldsymbol{w}\cdot\boldsymbol{D})dt$$

相加得：和 ρ^X 相关的项：

$$i_\rho d\gamma(\boldsymbol{X})=-\boldsymbol{w}\cdot d\boldsymbol{X}-\rho^X\times[\nabla\times(u\boldsymbol{b}+\boldsymbol{D})]\cdot d\boldsymbol{X}+\boldsymbol{w}\cdot\boldsymbol{D}dt+\rho^X\cdot[-\partial_t(u\boldsymbol{b}+\boldsymbol{D})-\nabla(\frac{u^2+w^2+D^2}{2})]dt$$
$$-\rho^X\times(\nabla\times\boldsymbol{w})\cdot d\boldsymbol{X}-\rho^X\cdot\partial_t\boldsymbol{w}dt+2\mu d\theta-\rho^X\cdot\nabla(\boldsymbol{w}\cdot\boldsymbol{D})dt$$
$$=-\rho^X\times[\nabla\times(u\boldsymbol{b}+\boldsymbol{D}+\boldsymbol{w})]\cdot d\boldsymbol{X}+\rho^X\cdot[-\partial_t(u\boldsymbol{b}+\boldsymbol{D}+\boldsymbol{w})-\nabla(\frac{u^2+w^2+D^2}{2}+\boldsymbol{w}\cdot\boldsymbol{D})]dt$$
$$-\boldsymbol{w}\cdot d\boldsymbol{X}+\boldsymbol{w}\cdot\boldsymbol{D}dt+2\mu d\theta$$

令 $M=[i_\rho d\gamma(\boldsymbol{X})]$

逐个计算 $d[i_\rho d\gamma(\boldsymbol{X})]$：

$$d\{-\rho^X\times[\nabla\times(u\boldsymbol{b}+\boldsymbol{D}+\boldsymbol{w})]\cdot d\boldsymbol{X}\}$$
$$=d(\boldsymbol{A}^{**}\cdot d\boldsymbol{X})$$
$$=-\varepsilon^i_{jk}(\nabla\times\boldsymbol{A}^{**})_i dX^j\wedge dX^k+\partial_t\boldsymbol{A}^{**}dt\wedge d\boldsymbol{X}+\partial_u\boldsymbol{A}^{**}du\wedge d\boldsymbol{X}+\partial_\theta\boldsymbol{A}^{**}d\theta\wedge d\boldsymbol{X}$$

其中 $\boldsymbol{A}^{**}=-\rho^X\times[\nabla\times(u\boldsymbol{b}+\boldsymbol{D}+\boldsymbol{w})]$

$$d\{\rho^X\cdot[-\partial_t(u\boldsymbol{b}+\boldsymbol{D}+\boldsymbol{w})-\nabla(\frac{u^2+w^2+D^2}{2}+\boldsymbol{w}\cdot\boldsymbol{D})]dt\}$$
$$=d(\phi^{**}dt)$$
$$=\nabla\phi^{**}d\boldsymbol{X}\wedge dt+\ldots$$

其中 $\phi^{**}=\rho^X\cdot[-\partial_t(u\boldsymbol{b}+\boldsymbol{D}+\boldsymbol{w})-\nabla(\frac{u^2+w^2+D^2}{2}+\boldsymbol{w}\cdot\boldsymbol{D})$

没有上面这两项是因为已经有 ρ^X，再乘以 $i_\rho d$ 就有 $\rho^X\rho^X$，会变成更高阶的量。

省略号表示的是不含 $d\boldsymbol{X}$ 的项，在 $\rho^X\partial_X$ 之后都会被舍去

$$d(-\boldsymbol{w}\cdot d\boldsymbol{X}+\boldsymbol{w}\cdot\boldsymbol{D}dt+2\mu d\theta)$$
$$=-\varepsilon^i_{jk}(\nabla\times(-\boldsymbol{w}))_i dX^j\wedge dX^k+\partial_t(-\boldsymbol{w})dt\wedge d\boldsymbol{X}+\partial_\mu(-\boldsymbol{w})d\mu\wedge d\boldsymbol{X}+\partial_\theta(-\boldsymbol{w})d\theta\wedge d\boldsymbol{X}$$
$$+\nabla(\boldsymbol{w}\cdot\boldsymbol{D})d\boldsymbol{X}\wedge dt+\ldots$$

$$
\begin{aligned}
& i_\rho d(-\boldsymbol{w}\cdot d\boldsymbol{X}+\boldsymbol{w}\cdot\boldsymbol{D}dt+2\mu d\theta) \\
& =-\rho^X\times(\nabla\times(-\boldsymbol{w}))\cdot d\boldsymbol{X}-\rho^X\cdot\partial_t(-\boldsymbol{w})dt-\rho^X\cdot\partial_\mu(-\boldsymbol{w})d\mu-\rho^X\cdot\partial_\theta(-\boldsymbol{w})d\theta \\
& +\rho^X\cdot\nabla(\boldsymbol{w}\cdot\boldsymbol{D})dt \\
& =\rho^X\times(\nabla\times\boldsymbol{w})\cdot d\boldsymbol{X}+\rho^X\cdot\partial_t\boldsymbol{w}dt+\rho^X\cdot\partial_\mu\boldsymbol{w}d\mu+\rho^X\cdot\partial_\theta\boldsymbol{w}d\theta+\rho^X\cdot\nabla(\boldsymbol{w}\cdot\boldsymbol{D})dt \\
& =\rho^X\times(\nabla\times\boldsymbol{w})\cdot d\boldsymbol{X}+\rho^X\cdot\partial_t\boldsymbol{w}dt+\rho^X\cdot\sqrt{\frac{B_0}{2\mu}}\boldsymbol{c}d\mu+\rho^X\cdot(-\sqrt{2\mu B_0}\boldsymbol{a})d\theta+\rho^X\cdot\nabla(\boldsymbol{w}\cdot\boldsymbol{D})dt \\
& =\rho^X\times(\nabla\times\boldsymbol{w})\cdot d\boldsymbol{X}+\rho^X\cdot\partial_t\boldsymbol{w}dt+\rho^X\cdot\nabla(\boldsymbol{w}\cdot\boldsymbol{D})dt-2\mu d\theta
\end{aligned}
$$

其中

$\frac{\varepsilon^2}{2}i_\rho d[i_\rho d\gamma(\boldsymbol{X})]$ 前面有个 $\frac{1}{2}$

$$\gamma(\boldsymbol{x})=\gamma(\boldsymbol{X})+\varepsilon i_\rho d\gamma(\boldsymbol{X})+\frac{\varepsilon^2}{2}i_\rho d[i_\rho d\gamma(\boldsymbol{X})]+...$$

$$=\boldsymbol{A}_0^*\cdot d\boldsymbol{X}-\phi_0^*dt+\boldsymbol{w}\cdot d\boldsymbol{X}-\boldsymbol{w}\cdot\boldsymbol{D}dt$$

$$-\rho^X\times[\nabla\times(u\boldsymbol{b}+\boldsymbol{D})]\cdot d\boldsymbol{X}-\boldsymbol{w}\cdot d\boldsymbol{X}+\boldsymbol{w}\cdot\boldsymbol{D}dt+\rho^X\cdot[-\partial_t(u\boldsymbol{b}+\boldsymbol{D})-\nabla(\frac{u^2+w^2+D^2}{2})]dt$$

$$-\rho^X\times(\nabla\times\boldsymbol{w})\cdot d\boldsymbol{X}-\rho^X\cdot\partial_t\boldsymbol{w}dt+2\mu d\theta-\rho^X\cdot\nabla(\boldsymbol{w}\cdot\boldsymbol{D})dt$$

$$+\frac{1}{2}\rho^X\times(\nabla\times\boldsymbol{w})\cdot d\boldsymbol{X}+\frac{1}{2}\rho^X\cdot\partial_t\boldsymbol{w}dt+\frac{1}{2}\rho^X\cdot\nabla(\boldsymbol{w}\cdot\boldsymbol{D})dt-\mu d\theta$$

$$=\boldsymbol{A}_0^*\cdot d\boldsymbol{X}-\phi_0^*dt$$

$$-\rho^X\times[\nabla\times(u\boldsymbol{b}+\boldsymbol{D})]\cdot d\boldsymbol{X}+\rho^X\cdot[-\partial_t(u\boldsymbol{b}+\boldsymbol{D})-\nabla(\frac{u^2+w^2+D^2}{2})]dt$$

$$-\rho^X\times(\nabla\times\frac{\boldsymbol{w}}{2})\cdot d\boldsymbol{X}-\rho^X\cdot\partial_t\frac{\boldsymbol{w}}{2}dt+\mu d\theta-\rho^X\cdot\nabla(\frac{\boldsymbol{w}\cdot\boldsymbol{D}}{2})dt$$

$$=\boldsymbol{A}_0^*\cdot d\boldsymbol{X}-\phi_0^*dt+\mu d\theta$$

$$-\rho^X\times[\nabla\times(u\boldsymbol{b}+\boldsymbol{D}+\frac{\boldsymbol{w}}{2})]\cdot d\boldsymbol{X}+\rho^X\cdot[-\partial_t(u\boldsymbol{b}+\boldsymbol{D}+\frac{\boldsymbol{w}}{2})-\nabla(\frac{u^2+w^2+D^2+\boldsymbol{w}\cdot\boldsymbol{D}}{2})]dt$$

得出$\gamma(\boldsymbol{x})=\gamma_0+\varepsilon\gamma_1+o(\varepsilon^2)...$

$$\gamma_0=\boldsymbol{A}_0^*\cdot d\boldsymbol{X}-\phi_0^*dt+\mu d\theta$$

$$\gamma_1=-\rho^X\times[\nabla\times\boldsymbol{A}_1^*]\cdot d\boldsymbol{X}+\rho^X\cdot[-\partial_t\boldsymbol{A}_1^*-\nabla\phi_1^*]dt$$
$$+\boldsymbol{A}_1(t,\boldsymbol{X}+\boldsymbol{\rho})\cdot d(\boldsymbol{X}+\boldsymbol{\rho})-\phi_1(t,\boldsymbol{X}+\boldsymbol{\rho})dt$$

$$\boldsymbol{A}_1^*=u\boldsymbol{b}+\boldsymbol{D}+\frac{\boldsymbol{w}}{2},$$
$$\phi_1^*=\frac{u^2+w^2+D^2+\boldsymbol{w}\cdot\boldsymbol{D}}{2}$$

γ_0并不是全是零阶，多阶次相加，由最低阶决定。

其中

$$\boldsymbol{a}=\boldsymbol{a}(\varepsilon\boldsymbol{X},\varepsilon t,\theta)$$

考虑量级时只管前面的ε，不管自变量的ε。

2、未扰动轨道的求解

未扰动轨道满足

$$i_\tau d\gamma_0 = 0$$

$$d\gamma_0 = -\varepsilon^i_{jk} B^*_{0i} \cdot dX^j \wedge dX^k + \boldsymbol{E}^*_0 d\boldsymbol{X} \wedge dt$$
$$+\boldsymbol{b} du \wedge d\boldsymbol{X} - u du \wedge dt - B_0 d\mu \wedge dt + d\mu \wedge d\theta$$

$$i_\tau d\gamma_0 = -\tau^{\boldsymbol{X}} \times \boldsymbol{B}^*_0 \cdot d\boldsymbol{X} + \tau^{\boldsymbol{X}} \cdot \boldsymbol{E}^*_0 dt - \tau^{\boldsymbol{X}} \cdot \boldsymbol{b} du$$
$$-\tau^t \boldsymbol{E}^*_0 \cdot d\boldsymbol{X} + \tau^u \boldsymbol{b} \cdot d\boldsymbol{X} - \tau^u u dt + \tau^t u du - \tau^\mu B_0 dt + \tau^t B_0 d\mu + \tau^\mu d\theta - \tau^\theta d\mu$$
$$= (-\tau^{\boldsymbol{X}} \times \boldsymbol{B}^*_0 - \tau^t \boldsymbol{E}^*_0 + \tau^u \boldsymbol{b}) \cdot d\boldsymbol{X} + (\tau^{\boldsymbol{X}} \cdot \boldsymbol{E}^*_0 - \tau^u u - \tau^\mu B_0) dt$$
$$+(\tau^t u - \tau^{\boldsymbol{X}} \cdot \boldsymbol{b}) du + (\tau^t B_0 - \tau^\theta) d\mu + \tau^\mu d\theta$$

$$-\frac{\tau^{\boldsymbol{X}}}{\tau^t} \times \boldsymbol{B}^*_0 - \boldsymbol{E}^*_0 + \frac{\tau^u}{\tau^t} \boldsymbol{b} = 0$$
$$\frac{\tau^{\boldsymbol{X}}}{\tau^t} \cdot \boldsymbol{E}^*_0 - \frac{\tau^u}{\tau^t} u = 0$$
$$u - \frac{\tau^{\boldsymbol{X}}}{\tau^t} \cdot \boldsymbol{b} = 0$$

得出未扰动轨道：

$$\frac{\tau^{\boldsymbol{X}}}{\tau^t} = \frac{u\boldsymbol{B}^*_0 + \boldsymbol{E}^*_0 \times \boldsymbol{b}}{B^*_0} = \boldsymbol{V}^*_c$$
$$\frac{\tau^u}{\tau^t} = \frac{\boldsymbol{E}^*_0 \cdot \boldsymbol{B}^*_0}{B^*_0}$$
$$\frac{\tau^\theta}{\tau^t} = B_0,$$
$$\frac{\tau^\mu}{\tau^t} = 0,$$

3、G 的求解

$$\gamma_1 + i_G d\gamma_0 + dS = 0$$
$$i_G d\gamma_0 = -\gamma_1 - dS$$

$$i_G d\gamma_0 = (-G^X \times \boldsymbol{B}_0^* - G^t \boldsymbol{E}_0^* + G^u \boldsymbol{b}) \cdot d\boldsymbol{X} + (G^X \cdot \boldsymbol{E}_0^* - G^u u - G^\mu B_0)dt$$
$$+(G^t u - G^X \cdot \boldsymbol{b})du + (G^t B_0 - G^\theta)d\mu + G^\mu d\theta$$

$$-\gamma_1 - dS$$
$$= \rho^X \times (\nabla \times \boldsymbol{A}_1^*) \cdot d\boldsymbol{X} - \rho^X \cdot (-\partial_t \boldsymbol{A}_1^* - \nabla \phi_1^*)dt$$
$$-\boldsymbol{A}_1 \cdot d\boldsymbol{X} - \boldsymbol{A}_1 \cdot d\boldsymbol{\rho} + \phi_1 dt$$
$$-\nabla S \cdot d\boldsymbol{X} - \partial_t S dt - \partial_u S du - \partial_\mu S d\mu - \partial_\theta S d\theta$$
$$= [\rho^X \times (\nabla \times \boldsymbol{A}_1^*) - \boldsymbol{A}_1 - \nabla S] \cdot d\boldsymbol{X} + [-\rho^X \cdot (-\partial_t \boldsymbol{A}_1^* - \nabla \phi_1^*) + \phi_1 - \partial_t S]dt$$
$$-\partial_u S du + (-\partial_\mu S - \boldsymbol{A}_1 \cdot \sqrt{\frac{1}{2B_0\mu}}\boldsymbol{a})d\mu + (-\partial_\theta S - \boldsymbol{A}_1 \cdot \sqrt{\frac{2\mu}{B_0}}\boldsymbol{c})d\theta$$
$$-\boldsymbol{A}_1 \cdot \partial_X(\sqrt{\frac{2\mu}{B_0}}\boldsymbol{a})d\boldsymbol{X} - \boldsymbol{A}_1 \cdot \partial_t(\sqrt{\frac{2\mu}{B_0}}\boldsymbol{a})dt$$

其中有

$$d\boldsymbol{\rho} = d\sqrt{\frac{2\mu}{B_0}}\boldsymbol{a} = \partial_X \boldsymbol{\rho} d\boldsymbol{X} + \partial_t \boldsymbol{\rho} dt + \partial_\mu \boldsymbol{\rho} d\mu + \partial_\theta \boldsymbol{\rho} d\theta$$
$$= \partial_X(\sqrt{\frac{2\mu}{B_0}}\boldsymbol{a})d\boldsymbol{X} + \partial_t(\sqrt{\frac{2\mu}{B_0}}\boldsymbol{a})dt + \sqrt{\frac{1}{2B_0\mu}}\boldsymbol{a}d\mu + \sqrt{\frac{2\mu}{B_0}}\boldsymbol{c}d\theta$$

由于 $-\gamma_1 - dS$ 的最后两项 $-\boldsymbol{A}_1 \cdot \partial_X(\sqrt{\frac{2\mu}{B_0}}\boldsymbol{a})d\boldsymbol{X} - \boldsymbol{A}_1 \cdot \partial_t(\sqrt{\frac{2\mu}{B_0}}\boldsymbol{a})dt$ 包含

$-\boldsymbol{A}_1 \cdot \partial_X \boldsymbol{a}$ 相乘为二阶量，舍去

$\boldsymbol{A}_1$ 是扰动电磁场，$\partial_X \boldsymbol{a}$ 是本底的扰动，若是两者相乘不能省略则需要考虑——参量不稳定

令 $G^t = 0$

$$-G^X \times \boldsymbol{B}_0^* + G^u \boldsymbol{b} = \rho^X \times (\nabla \times \boldsymbol{A}_1^*) - \boldsymbol{A}_1 - \nabla S,$$
$$-G^X \cdot \boldsymbol{b} = -\partial_u S,$$
$$-G^\theta = -\partial_\mu S - \boldsymbol{A}_1 \cdot \sqrt{\frac{1}{2B_0\mu}}\boldsymbol{a},$$
$$G^\mu = -\partial_\theta S - \boldsymbol{A}_1 \cdot \sqrt{\frac{2\mu}{B_0}}\boldsymbol{c}$$

得出扰动场：

$$G^X = \frac{1}{B_0^*}(\rho^X \times \boldsymbol{B}_1^* - \boldsymbol{A}_1 - \nabla S) \times \boldsymbol{b} + \frac{\boldsymbol{B}_0^*}{B_0^*}\partial_u S$$
$$G^u = \frac{\boldsymbol{B}_0^*}{B_0^*} \cdot (\rho^X \times \boldsymbol{B}_1^* - \boldsymbol{A}_1 - \nabla S)$$
$$G^\theta = \partial_\mu S + \boldsymbol{A}_1 \cdot \sqrt{\frac{1}{2B_0\mu}}\boldsymbol{a},$$
$$G^\mu = -\partial_\theta S - \boldsymbol{A}_1 \cdot \sqrt{\frac{2\mu}{B_0}}\boldsymbol{c}$$

由于 S 定义的正负不同，和文献中正负号不一样

4、S 的求解 1：无穷项贝塞尔函数求和形式

定义 $S \equiv -H_1 + i_G \gamma_0$

$i_\tau dS = -i_\tau \gamma_1$

$S|_{\lambda_0} = -\int_{\lambda_0} [\boldsymbol{A}_1(t, \boldsymbol{X}+\boldsymbol{\rho}) \cdot d(\boldsymbol{X}+\boldsymbol{\rho}) - \phi_1(t, \boldsymbol{X}+\boldsymbol{\rho}) dt]$

其中舍掉的 $-\rho^X \times (\nabla \times \boldsymbol{A}_1^*) \cdot d\boldsymbol{X} + \rho^X \cdot (-\partial_t \boldsymbol{A}_1^* - \nabla \phi_1^*) dt$ 是不考虑 $\boldsymbol{A}_1^*$，ϕ_1^* 的不均匀性

由未扰动轨道代入得：

$S|_{\lambda_0} = -\int_{\lambda_0} [\boldsymbol{A}_1(t, \boldsymbol{X}+\boldsymbol{\rho}) \cdot d(\boldsymbol{X}+\boldsymbol{\rho}) - \phi_1(t, \boldsymbol{X}+\boldsymbol{\rho}) dt]$

$= \int_{\lambda_0} [\phi_1(t, \boldsymbol{X}+\boldsymbol{\rho}) dt - \boldsymbol{A}_1(t, \boldsymbol{X}+\boldsymbol{\rho}) \cdot (\boldsymbol{V}_c^* + \boldsymbol{w}) dt]$

即

$S(\boldsymbol{X}, u, \mu, \theta, t)$

$= \int_{-\infty}^{t} [\phi_1(t', \boldsymbol{X}'+\boldsymbol{\rho}') dt' - \boldsymbol{A}_1(t', \boldsymbol{X}'+\boldsymbol{\rho}') \cdot (\boldsymbol{V}_c^{*'} + \boldsymbol{w}') dt']$

设 $\boldsymbol{A}_1(t, \boldsymbol{X}+\boldsymbol{\rho}) = \boldsymbol{A}_1 e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} = \boldsymbol{A}_1 e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{X} + i\boldsymbol{k}\cdot\boldsymbol{\rho}}$

以及

$\boldsymbol{a} = \boldsymbol{e}_1 \cos\theta - \boldsymbol{e}_2 \sin\theta$

$\boldsymbol{c} = -\boldsymbol{e}_1 \sin\theta - \boldsymbol{e}_2 \cos\theta$

$\boldsymbol{\rho} = \sqrt{\frac{2\mu}{B_0}} \boldsymbol{a} = \sqrt{\frac{2\mu}{B_0}} (\boldsymbol{e}_1 \cos\theta - \boldsymbol{e}_2 \sin\theta)$

$\boldsymbol{w} = \sqrt{2\mu B_0} \boldsymbol{c} = \sqrt{2\mu B_0} (-\boldsymbol{e}_1 \sin\theta - \boldsymbol{e}_2 \cos\theta)$

$\boldsymbol{X}, u, \mu, \theta$ 的变化和 t 有关：

假设均匀恒定零阶磁场，零阶电场为 0：

$\boldsymbol{B}_0 = B_0 \boldsymbol{b}, \boldsymbol{E}_0 = 0,$

$\boldsymbol{B}_0^* = \nabla \times \boldsymbol{A}_0^* = B_0 \boldsymbol{b} + \nabla \times (u\boldsymbol{b} + \boldsymbol{D}) = B_0 \boldsymbol{b}$

$\boldsymbol{E}_0^* = \boldsymbol{E}_0 - \partial_t (u\boldsymbol{b} + \boldsymbol{D}) - \nabla \frac{u^2 + w^2 + D^2}{2} = 0$

$B_0, u\boldsymbol{b}, \boldsymbol{D}, \mu, \boldsymbol{k}$ 不随着时间变？， $\boldsymbol{V}_c^{*'} = \boldsymbol{V}_c' = u\boldsymbol{b} + \boldsymbol{D}$ 也不变。只有 $\boldsymbol{\rho}, \boldsymbol{w}$ 在变

即：

$$\frac{\tau^X}{\tau^t}=\frac{u\boldsymbol{B}_0^*+\boldsymbol{E}_0^*\times\boldsymbol{b}}{B_0^*}=\boldsymbol{V}_c\ =u\boldsymbol{b}+\boldsymbol{D}$$

$$\frac{\tau^u}{\tau^t}=\frac{\boldsymbol{E}_0^*\cdot\boldsymbol{B}_0^*}{B_0^*}=0$$

$$\frac{\tau^\theta}{\tau^t}=B_0,$$

$$\frac{\tau^\mu}{\tau^t}=0,$$

令

$$\tau\equiv t-t',$$

$$t'=t-\tau$$

$$\theta=\theta'+\tau B_0,$$

$$\theta'=\theta-\tau B_0$$

$$\boldsymbol{V}_c'=\boldsymbol{V}_c$$

$$\boldsymbol{X}'-\boldsymbol{X}=(t'-t)\,\boldsymbol{V}_c\ ,$$

$$\boldsymbol{X}'=\boldsymbol{X}-\tau\boldsymbol{V}_c$$

$$\boldsymbol{\rho}'=\sqrt{\frac{2\mu}{B_0}}(\boldsymbol{e}_1\cos\theta'-\boldsymbol{e}_2\sin\theta')=\sqrt{\frac{2\mu}{B_0}}(\boldsymbol{e}_1\cos\left(\theta-\tau B_0\right)-\boldsymbol{e}_2\sin\left(\theta-\tau B_0\right))$$

$$\boldsymbol{w}'=\sqrt{2\mu B_0}(-\boldsymbol{e}_1\sin\theta'-\boldsymbol{e}_2\cos\theta')=\sqrt{2\mu B_0}[-\boldsymbol{e}_1\sin(\theta-\tau B_0)-\boldsymbol{e}_2\cos(\theta-\tau B_0)]$$

考虑回旋运动的回旋角的定义，就会有正负号的差别

$$S(\boldsymbol{X},u,\mu,\theta,t)$$

$$=\int_{-\infty}^{t}[\phi_1(t',\boldsymbol{X}'+\boldsymbol{\rho})dt'-\boldsymbol{A}_1(t',\boldsymbol{X}'+\boldsymbol{\rho})\cdot(\boldsymbol{V}_c^{*'}+\boldsymbol{w}')dt']$$

$$=\int_{-\infty}^{t}e^{-i\omega t'+i\boldsymbol{k}'\cdot\boldsymbol{X}'+i\boldsymbol{k}'\cdot\boldsymbol{\rho}'}[\phi_1-\boldsymbol{A}_1\cdot(\boldsymbol{V}_c^{*'}+\boldsymbol{w}')]dt'$$

$$=\int_0^\infty \exp[-i\omega\left(t-\tau\right)+i\boldsymbol{k}\cdot\left(\boldsymbol{X}-\tau\boldsymbol{V}_c\right)+i\boldsymbol{k}\cdot\left[\sqrt{\frac{2\mu}{B_0}}(\boldsymbol{e}_1\cos\left(\theta-\tau B_0\right)-\boldsymbol{e}_2\sin\left(\theta-\tau B_0\right))\right]]$$
$$\{\phi_1-\boldsymbol{A}_1\cdot(\boldsymbol{V}_c+\left[\sqrt{2\mu B_0}[-\boldsymbol{e}_1\sin(\theta-\tau B_0)-\boldsymbol{e}_2\cos(\theta-\tau B_0)]\right])\}d\tau$$

$\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{b}$ 和 $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ 是两种坐标系。现在用 $\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{b}$，如果 b 不变，该坐标系是不变的。

假设 $\boldsymbol{k}=k_1\boldsymbol{e}_1+k_{//}\boldsymbol{b}$，即 $\boldsymbol{k}$ 垂直方向只有 $\boldsymbol{e}_1$ 分量，可化简

$$
\begin{aligned}
&S(\boldsymbol{X},u,\mu,\theta,t)\\
&=\int_0^\infty \exp[-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}+i\omega\tau-i\boldsymbol{k}\cdot\boldsymbol{V}_c\tau+ik_1\sqrt{\frac{2\mu}{B_0}}\cos\left(\theta-\tau B_0\right)]\\
&\quad\{\phi_1-\boldsymbol{A}_1\cdot\boldsymbol{V}_c+\sqrt{2\mu B_0}\left[\boldsymbol{A}_1\cdot\boldsymbol{e}_1\sin(\theta-\tau B_0)+\boldsymbol{A}_1\cdot\boldsymbol{e}_2\cos(\theta-\tau B_0)\right]\}d\tau\\
&=e^{\left(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}\right)}\int_0^\infty e^{\beta}\{\phi_1-\boldsymbol{A}_1\cdot\boldsymbol{V}_c+\sqrt{2\mu B_0}\left[\boldsymbol{A}_1\cdot\boldsymbol{e}_1\sin(\theta-\tau B_0)+\boldsymbol{A}_1\cdot\boldsymbol{e}_2\cos(\theta-\tau B_0)\right]\}d\tau\\
&=e^{\left(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}\right)}\int_0^\infty e^{\beta}\{\phi_1-\boldsymbol{A}_1\cdot\boldsymbol{V}_c+\sqrt{2\mu B_0}A_{11}\sin(\theta-\tau B_0)+\sqrt{2\mu B_0}A_{12}\cos(\theta-\tau B_0)\}d\tau\\
&=e^{\left(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}\right)}\left[\int_0^\infty e^{\beta}\left(\phi_1-\boldsymbol{A}_1\cdot\boldsymbol{V}_c\right)d\tau+\sqrt{2\mu B_0}\int_0^\infty e^{\beta}\left[A_{11}\sin(\theta-\tau B_0)+A_{12}\cos(\theta-\tau B_0)\right]d\tau\right]\\
&=e^{\left(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}\right)}\left[\left(\phi_1-\boldsymbol{A}_1\cdot\boldsymbol{V}_c\right)\int_0^\infty e^{\beta}d\tau+\sqrt{2\mu B_0}A_{11}\int_0^\infty e^{\beta}\sin(\theta-\tau B_0)d\tau+\sqrt{2\mu B_0}A_{12}\int_0^\infty e^{\beta}\cos(\theta-\tau B_0)d\tau\right]
\end{aligned}
$$

令 $\beta=i\omega\tau-i\boldsymbol{k}\cdot\boldsymbol{V}_c\tau+ik_1\sqrt{\frac{2\mu}{B_0}}\cos\left(\theta-\tau B_0\right)$

令 $\rho_0=\sqrt{\frac{2\mu}{B_0}}$， $\lambda=k_1\sqrt{\frac{2\mu}{B_0}}=k_1\rho_0$，

则 $\beta=\left(i\omega-i\boldsymbol{k}\cdot\boldsymbol{V}_c\right)\tau+i\lambda\cos\left(\theta-\tau B_0\right)$

其中需要求解的有：$\int_0^\infty e^{\beta}d\tau$ 和 $\int_0^\infty e^{\beta}\sin(\theta-\tau B_0)d\tau$ 和 $\int_0^\infty e^{\beta}\cos(\theta-\tau B_0)d\tau$

$$
\begin{aligned}
&\int_0^\infty e^{\beta}d\tau=\int_0^\infty \exp\left[\left(i\omega-i\boldsymbol{k}\cdot\boldsymbol{V}_c\right)\tau+i\lambda\cos\left(\theta-\tau B_0\right)\right]d\tau\\
&=\int_0^\infty \exp\left[\left(i\omega-i\boldsymbol{k}\cdot\boldsymbol{V}_c\right)\tau\right]\exp\left[i\lambda\cos\left(\theta-\tau B_0\right)\right]d\tau\\
&=\int_0^\infty \exp\left[\left(i\omega-i\boldsymbol{k}\cdot\boldsymbol{V}_c\right)\tau\right]\sum_{n=-\infty}^{n=\infty}I_n\left(i\lambda\right)e^{in\left(\theta-\tau B_0\right)}d\tau\\
&=\sum_{n=-\infty}^{n=\infty}I_n\left(i\lambda\right)e^{in\theta}\int_0^\infty \exp\left[-i\left(nB_0+\boldsymbol{k}\cdot\boldsymbol{V}_c-\omega\right)\tau\right]d\tau\\
&=\sum_{n=-\infty}^{n=\infty}\frac{I_n\left(i\lambda\right)e^{in\theta}}{i\left(nB_0+\boldsymbol{k}\cdot\boldsymbol{V}_c-\omega\right)}
\end{aligned}
$$

利用 $\exp\left[i\lambda\cos\left(\theta-\tau B_0\right)\right]=\sum_{n=-\infty}^{n=\infty}I_n\left(i\lambda\right)e^{in\left(\theta-\tau B_0\right)}$

$$\int_0^\infty e^\beta \sin(\theta-\tau B_0)d\tau$$
$$=\int_0^\infty \exp\left[\left(i\omega-i\boldsymbol{k}\cdot\boldsymbol{V}_c\right)\tau+i\lambda\cos\left(\theta-\tau B_0\right)\right]\sin(\theta-\tau B_0)d\tau$$
$$=\int_0^\infty \exp\left[\left(i\omega-i\boldsymbol{k}\cdot\boldsymbol{V}_c\right)\tau\right]\exp\left[i\lambda\cos\left(\theta-\tau B_0\right)\right]\sin(\theta-\tau B_0)d\tau$$
$$=\int_0^\infty \exp\left[\left(i\omega-i\boldsymbol{k}\cdot\boldsymbol{V}_c\right)\tau\right]\sum_{n=-\infty}^{n=\infty}\frac{-nI_n\left(i\lambda\right)}{\lambda}e^{in\left(\theta-\tau B_0\right)}d\tau$$
$$=\sum_{n=-\infty}^{n=\infty}\frac{-nI_n\left(i\lambda\right)}{\lambda}e^{in\theta}\int_0^\infty \exp\left[-i\left(nB_0+\boldsymbol{k}\cdot\boldsymbol{V}_c-\omega\right)\tau\right]d\tau$$
$$=\sum_{n=-\infty}^{n=\infty}\frac{nI_n\left(i\lambda\right)e^{in\theta}}{-i\lambda\left(nB_0+\boldsymbol{k}\cdot\boldsymbol{V}_c-\omega\right)}$$

利用 $\exp\left[i\lambda\cos\left(\theta-\tau B_0\right)\right]\sin(\theta-\tau B_0)=\sum_{n=-\infty}^{n=\infty}\frac{-nI_n\left(i\lambda\right)}{\lambda}e^{in\left(\theta-\tau B_0\right)}$

$$\int_0^\infty e^\beta \cos(\theta-\tau B_0)d\tau$$
$$=\int_0^\infty \exp\left[\left(i\omega-i\boldsymbol{k}\cdot\boldsymbol{V}_c\right)\tau+i\lambda\cos\left(\theta-\tau B_0\right)\right]\cos(\theta-\tau B_0)d\tau$$
$$=\int_0^\infty \exp\left[\left(i\omega-i\boldsymbol{k}\cdot\boldsymbol{V}_c\right)\tau\right]\exp\left[i\lambda\cos\left(\theta-\tau B_0\right)\right]\cos(\theta-\tau B_0)d\tau$$
$$=\int_0^\infty \exp\left[\left(i\omega-i\boldsymbol{k}\cdot\boldsymbol{V}_c\right)\tau\right]\sum_{n=-\infty}^{n=\infty}I_n^{'}\left(i\lambda\right)e^{in\left(\theta-\tau B_0\right)}d\tau$$
$$=\sum_{n=-\infty}^{n=\infty}I_n^{'}\left(i\lambda\right)e^{in\theta}\int_0^\infty \exp\left[-i\left(nB_0+\boldsymbol{k}\cdot\boldsymbol{V}_c-\omega\right)\tau\right]d\tau$$
$$=\sum_{n=-\infty}^{n=\infty}\frac{I_n^{'}\left(i\lambda\right)e^{in\theta}}{i\left(nB_0+\boldsymbol{k}\cdot\boldsymbol{V}_c-\omega\right)}$$

利用 $\exp\left[i\lambda\cos\left(\theta-\tau B_0\right)\right]\cos(\theta-\tau B_0)=\sum_{n=-\infty}^{n=\infty}I_n^{'}\left(i\lambda\right)e^{in\left(\theta-\tau B_0\right)}$

$$
\begin{aligned}
&S(\boldsymbol{X},u,\mu,\theta,t)\\
&=e^{(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X})}\left[\left(\phi_1-\boldsymbol{A}_1\cdot\boldsymbol{V}_c\right)\int_0^\infty e^{\beta}d\tau+\sqrt{2\mu B_0}A_{11}\int_0^\infty e^{\beta}\sin(\theta-\tau B_0)d\tau+\sqrt{2\mu B_0}A_{12}\int_0^\infty e^{\beta}\cos(\theta-\tau B_0)d\tau\right]\\
&=e^{(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X})}[\left(\phi_1-\boldsymbol{A}_1\cdot\boldsymbol{V}_c\right)\sum_{n=-\infty}^{n=\infty}\frac{I_n(i\lambda)e^{in\theta}}{i\left(nB_0+\boldsymbol{k}\cdot\boldsymbol{V}_c-\omega\right)}\\
&+\sqrt{2\mu B_0}A_{11}\sum_{n=-\infty}^{n=\infty}\frac{nI_n(i\lambda)e^{in\theta}}{-i\lambda\left(nB_0+\boldsymbol{k}\cdot\boldsymbol{V}_c-\omega\right)}+\sqrt{2\mu B_0}A_{12}\sum_{n=-\infty}^{n=\infty}\frac{I_n'(i\lambda)e^{in\theta}}{i\left(nB_0+\boldsymbol{k}\cdot\boldsymbol{V}_c-\omega\right)}]\\
&=e^{(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X})}[\left(\phi_1-A_{1b}u\right)\sum_{n=-\infty}^{n=\infty}\frac{I_n(i\lambda)e^{in\theta}}{i\left(nB_0+k_bu-\omega\right)}\\
&+\sqrt{2\mu B_0}A_{11}\sum_{n=-\infty}^{n=\infty}\frac{nI_n(i\lambda)e^{in\theta}}{-i\lambda\left(nB_0+k_bu-\omega\right)}+\sqrt{2\mu B_0}A_{12}\sum_{n=-\infty}^{n=\infty}\frac{I_n'(i\lambda)e^{in\theta}}{i\left(nB_0+k_bu-\omega\right)}]\\
&=e^{(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X})}\sum_{n=-\infty}^{n=\infty}\frac{e^{in\theta}}{i\left(nB_0+k_bu-\omega\right)}[\left(\phi_1-A_{1b}u\right)I_n(i\lambda)+\sqrt{2\mu B_0}A_{11}\frac{nI_n(i\lambda)}{-\lambda}+\sqrt{2\mu B_0}A_{12}I_n'(i\lambda)]
\end{aligned}
$$

注意 $\lambda=k_1\sqrt{\frac{2\mu}{B_0}}=k_1\rho_0$ 与 μ 相关

由

$$
\begin{aligned}
&S(\boldsymbol{X},u,\mu,\theta,t)\\
&=e^{(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X})}\sum_{n=-\infty}^{n=\infty}\frac{e^{in\theta}}{i\left(nB_0+k_bu-\omega\right)}[\left(\phi_1-A_{1b}u\right)I_n(i\lambda)+\sqrt{2\mu B_0}A_{11}\frac{nI_n(i\lambda)}{-\lambda}+\sqrt{2\mu B_0}A_{12}I_n'(i\lambda)]
\end{aligned}
$$

得到：

$$
\begin{aligned}
&\nabla S(\boldsymbol{X},u,\mu,\theta,t)\\
&=i\boldsymbol{k}S\\
&=i\boldsymbol{k}e^{(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X})}\sum_{n=-\infty}^{n=\infty}\frac{e^{in\theta}}{i\left(nB_0+k_bu-\omega\right)}[\left(\phi_1-A_{1b}u\right)I_n(i\lambda)+\sqrt{2\mu B_0}A_{11}\frac{nI_n(i\lambda)}{-\lambda}+\sqrt{2\mu B_0}A_{12}I_n'(i\lambda)]
\end{aligned}
$$

$$
\begin{aligned}
&\partial_\theta S(\boldsymbol{X},u,\mu,\theta,t)\\
&=inS\\
&=ine^{(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X})}\sum_{n=-\infty}^{n=\infty}\frac{e^{in\theta}}{i\left(nB_0+k_bu-\omega\right)}[\left(\phi_1-A_{1b}u\right)I_n(i\lambda)+\sqrt{2\mu B_0}A_{11}\frac{nI_n(i\lambda)}{-\lambda}+\sqrt{2\mu B_0}A_{12}I_n'(i\lambda)]
\end{aligned}
$$

5、S 的求解 2：有限项贝塞尔函数形式

$$S(\boldsymbol{X},u,\mu,\theta,t)$$
$$=e^{(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X})}\left[\left(\phi_1-\boldsymbol{A}_1\cdot\boldsymbol{V}_c\right)\int_0^\infty e^\beta d\tau+\sqrt{2\mu B_0}A_{11}\int_0^\infty e^\beta\sin(\theta-\tau B_0)d\tau+\sqrt{2\mu B_0}A_{12}\int_0^\infty e^\beta\cos(\theta-\tau B_0)d\tau\right]$$

其中需要求解的有：$\int_0^\infty e^\beta d\tau$ 和 $\int_0^\infty e^\beta\sin(\theta-\tau B_0)d\tau$ 和 $\int_0^\infty e^\beta\cos(\theta-\tau B_0)d\tau$

$$\beta=i\omega\tau-i\boldsymbol{k}\cdot\boldsymbol{V}_c\,\tau+ik_1\sqrt{\frac{2\mu}{B_0}}\cos\left(\theta-\tau B_0\right)$$

令 $\tau B_0=s$， $a=\dfrac{\omega-\boldsymbol{k}\cdot\boldsymbol{V}_c}{B_0}$，

已经有 $\rho_0=\sqrt{\dfrac{2\mu}{B_0}}$， $\lambda=k_1\sqrt{\dfrac{2\mu}{B_0}}=k_1\rho_0$， 则有

$$\beta=i\omega\tau-i\boldsymbol{k}\cdot\boldsymbol{V}_c\,\tau+ik_1\sqrt{\frac{2\mu}{B_0}}\cos\left(\theta-\tau B_0\right)$$
$$=iaB_0\tau+i\lambda\cos\left(\theta-\tau B_0\right)$$
$$=ias+i\lambda\cos\left(\theta-s\right)$$

*引入 g 的定义：

$$g\left(\theta,\lambda\right)=\int_0^\infty e^\beta d\tau$$
$$=\int_0^\infty\exp\left[iaB_0\tau+i\lambda\cos\left(\theta-\tau B_0\right)\right]d\tau$$
$$=\frac{1}{B_0}\int_0^\infty\exp\left[ias+i\lambda\cos\left(\theta-s\right)\right]ds$$

则有：

$$\frac{\partial g\left(\theta,\lambda\right)}{\partial\theta}$$
$$=\frac{1}{B_0}\int_0^\infty\frac{\partial}{\partial\theta}\exp\left[ias+i\lambda\cos\left(\theta-s\right)\right]ds$$
$$=-\frac{i\lambda}{B_0}\int_0^\infty\exp\left[ias+i\lambda\cos\left(\theta-s\right)\right]\sin\left(\theta-s\right)ds$$

$$\int_0^\infty e^\beta \sin(\theta-\tau B_0)d\tau$$
$$=\frac{1}{B_0}\int_0^\infty \exp\left[ias+i\lambda\cos\left(\theta-s\right)\right]\sin(\theta-s)ds$$
$$=\frac{i}{\lambda}\frac{\partial g\left(\theta,\lambda\right)}{\partial\theta}$$

$$\frac{\partial g\left(\theta,\lambda\right)}{\partial\lambda}$$
$$=\frac{1}{B_0}\int_0^\infty \frac{\partial}{\partial\lambda}\exp\left[ias+i\lambda\cos\left(\theta-s\right)\right]ds$$
$$=\frac{i}{B_0}\int_0^\infty \exp\left[ias+i\lambda\cos\left(\theta-s\right)\right]\cos\left(\theta-s\right)ds$$

$$\int_0^\infty e^\beta \cos(\theta-\tau B_0)d\tau$$
$$=\frac{1}{B_0}\int_0^\infty \exp\left[ias+i\lambda\cos\left(\theta-s\right)\right]\cos(\theta-s)d\tau$$
$$=-i\frac{\partial g\left(\theta,\lambda\right)}{\partial\lambda}$$

$$g\left(\theta,\lambda\right)=g\left(\theta+2\pi,\lambda\right)$$
$$=\frac{1}{B_0}\int_0^\infty \exp\left[ias+i\lambda\cos\left(\theta+2\pi-s\right)\right]ds$$
$$=\frac{1}{B_0}\int_0^\infty e^{ia2\pi}\exp\left[ias-ia2\pi+i\lambda\cos\left(\theta+2\pi-s\right)\right]ds$$
$$=\frac{1}{B_0}\int_0^\infty e^{ia2\pi}\exp\left[ia(s-2\pi)+i\lambda\cos\left(\theta-(s-2\pi)\right)\right]ds$$
$$=e^{ia2\pi}\frac{1}{B_0}\int_{-2\pi}^\infty \exp\left[iat+i\lambda\cos\left(\theta-t\right)\right]dt$$
$$=e^{ia2\pi}\{\frac{1}{B_0}\int_{-2\pi}^0 \exp\left[iat+i\lambda\cos\left(\theta-t\right)\right]dt+\frac{1}{B_0}\int_0^\infty \exp\left[iat+i\lambda\cos\left(\theta-t\right)\right]dt\}$$
$$=e^{ia2\pi}\{\frac{1}{B_0}\int_{-2\pi}^0 \exp\left[iat+i\lambda\cos\left(\theta-t\right)\right]dt+g\}$$

即 $g=\frac{e^{ia2\pi}}{B_0}\int_{-2\pi}^0 \exp\left[iat+i\lambda\cos\left(\theta-t\right)\right]dt+e^{ia2\pi}g$

***不用无穷项求和而表示出的 g**

$$g=\frac{e^{ia2\pi}}{B_0\left(1-e^{ia2\pi}\right)}\int_{-2\pi}^{0}\exp\left[iat+i\lambda\cos(\theta-t)\right]dt$$
$$=\frac{C_0}{2\pi}\int_{-2\pi}^{0}\exp\left[iat+i\lambda\cos(\theta-t)\right]dt$$
$$=\frac{C_0}{2\pi}\int_{0}^{2\pi}\exp\left[-iat+i\lambda\cos(\theta+t)\right]dt$$

$$\frac{e^{ia2\pi}}{B_0\left(1-e^{ia2\pi}\right)}=\frac{C_0}{2\pi}$$

C_0？？

$$\frac{\partial g}{\partial\theta}$$
$$=\frac{C_0}{2\pi}\int_{0}^{2\pi}\frac{\partial}{\partial\theta}\exp\left[-iat+i\lambda\cos(\theta+t)\right]dt$$
$$=\frac{C_0}{2\pi}\int_{0}^{2\pi}\left[-i\lambda\sin(\theta+t)\right]\exp\left[-iat+i\lambda\cos(\theta+t)\right]dt$$
$$\frac{\partial g}{\partial\theta}(\theta)=\frac{\partial g}{\partial\theta}(\theta+2\pi)$$

$$\frac{\partial g}{\partial\lambda}=\frac{C_0}{2\pi}\int_{0}^{2\pi}i\cos(\theta+t)\exp\left[-iat+i\lambda\cos(\theta+t)\right]dt$$
$$\frac{\partial g}{\partial\lambda}(\theta)=\frac{\partial g}{\partial\lambda}(\theta+2\pi)$$

$$S(\boldsymbol{X},u,\mu,\theta,t)$$
$$=e^{(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X})}\left[\left(\phi_1-\boldsymbol{A}_1\cdot\boldsymbol{V}_c\right)g+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial g}{\partial\lambda}\right)\right]$$

则有：

$$\nabla S(\boldsymbol{X},u,\mu,\theta,t)$$
$$=i\boldsymbol{k}S$$
$$\partial_\theta S(\boldsymbol{X},u,\mu,\theta,t)$$
$$=e^{(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X})}\left[\left(\phi_1-\boldsymbol{A}_1\cdot\boldsymbol{V}_c\right)\partial_\theta g+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\partial_\theta\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{12}\partial_\theta\left(-i\frac{\partial g}{\partial\lambda}\right)\right]$$
$$=e^{(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X})}\left[\left(\phi_1-\boldsymbol{A}_1\cdot\boldsymbol{V}_c\right)\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial^2 g}{\partial\theta^2}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial^2 g}{\partial\theta\partial\lambda}\right)\right]$$

6、$L_G F$ 的求解

$L_G F$ 是分布函数的扰动，和经典的参量不稳定性里面 $f_{low} f_{side}$ 的区别在哪里？

首先有 $F(\boldsymbol{X},t,u,\mu)$ 是未扰动分布函数，而且是回旋中心的，与 θ 无关，满足 Vlasov 方程：

$$i_\tau dF = 0$$

设 $F = n_0 \exp\left[-\frac{\left(u^2+D^2\right)2+\mu B_0}{T}\right]$

由于 $\boldsymbol{E}_0 = 0, D = 0$

$$F = n_0 \exp\left(-\frac{u^2/2+\mu B_0}{T}\right)$$

此时的未扰动轨道：

$$\frac{\tau^X}{\tau^t} = \frac{u\boldsymbol{B}_0^* + \boldsymbol{E}_0^* \times \boldsymbol{b}}{B_0^*} = \boldsymbol{V}_c \;= u\boldsymbol{b} + \boldsymbol{D} = u\boldsymbol{b}$$

$$\frac{\tau^u}{\tau^t} = \frac{\boldsymbol{E}_0^* \cdot \boldsymbol{B}_0^*}{B_0^*} = 0$$

$$\frac{\tau^\theta}{\tau^t} = B_0,$$

$$\frac{\tau^\mu}{\tau^t} = 0,$$

$\nabla F = 0$，

$$\partial_u F = \partial_u n_0 \exp\left(-\frac{u^2/2+\mu B_0}{T}\right) = -\frac{u}{T} n_0 \exp\left(-\frac{u^2/2+\mu B_0}{T}\right)$$

$$\partial_\mu F = \partial_\mu n_0 \exp\left(-\frac{u^2/2+\mu B_0}{T}\right) = -\frac{B_0}{T} n_0 \exp\left(-\frac{u^2/2+\mu B_0}{T}\right)$$

则有：

$$L_G F = i_G dF$$

$$= G^X \cdot \nabla F + G^u \partial_u F + G^\mu \partial_\mu F$$

$$= G^u \partial_u F + G^\mu \partial_\mu F$$

$$G^u = \frac{\boldsymbol{B}_0^*}{B_0^*}\cdot(\rho^X \times \boldsymbol{B}_1^* - \boldsymbol{A}_1 - \nabla S) = \boldsymbol{b}\cdot(-\boldsymbol{A}_1 e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{X} + i\boldsymbol{k}\cdot\boldsymbol{\rho}} - i\boldsymbol{k}S) = -A_{1b}e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{X} + i\boldsymbol{k}\cdot\boldsymbol{\rho}} - ik_b S$$

其中 $\boldsymbol{B}_1^* = \nabla\times\boldsymbol{A}_1^* = \nabla\times\left(u\boldsymbol{b} + \boldsymbol{D} + \frac{\boldsymbol{w}}{2}\right) = 0$

$$G^\mu = -\partial_\theta S - \boldsymbol{A}_1 \cdot \sqrt{\frac{2\mu}{B_0}}\boldsymbol{c}$$
$$= -\partial_\theta S - e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{X} + i\boldsymbol{k}\cdot\boldsymbol{\rho}}\sqrt{\frac{2\mu}{B_0}}\boldsymbol{A}_1 \cdot\left(-\boldsymbol{e}_1\sin\theta - \boldsymbol{e}_2\cos\theta\right)$$
$$= -\partial_\theta S + e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{X} + i\boldsymbol{k}\cdot\boldsymbol{\rho}}\sqrt{\frac{2\mu}{B_0}}\left(A_{11}\sin\theta + A_{12}\cos\theta\right)$$

$$L_{\boldsymbol{G}}F(\boldsymbol{X},t,u,\mu,\theta)$$
$$= G^u\partial_u F + G^\mu\partial_\mu F$$
$$= \left(-A_{1b}e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{X} + i\boldsymbol{k}\cdot\boldsymbol{\rho}} - ik_b S\right)\partial_u F + \left[-\partial_\theta S + e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{X} + i\boldsymbol{k}\cdot\boldsymbol{\rho}}\rho_0\left(A_{11}\sin\theta + A_{12}\cos\theta\right)\right]\partial_\mu F$$

其中无穷项形式下：

$$S(\boldsymbol{X},u,\mu,\theta,t)$$
$$= e^{(-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{X})}\sum_{n=-\infty}^{n=\infty}\frac{e^{in\theta}}{i\left(nB_0 + k_b u - \omega\right)}[\left(\phi_1 - A_{1b}u\right)I_n\left(i\lambda\right) + \sqrt{2\mu B_0}A_{11}\frac{nI_n\left(i\lambda\right)}{-\lambda} + \sqrt{2\mu B_0}A_{12}I_n^{'}\left(i\lambda\right)]$$

有限项形式下：

$$S(\boldsymbol{X},u,\mu,\theta,t)$$
$$= e^{(-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{X})}\left[\left(\phi_1 - \boldsymbol{A}_1 \cdot\boldsymbol{V}_c\right)g + \sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial g}{\partial\theta} + \sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial g}{\partial\lambda}\right)\right]$$

7、J1 的求解 1：无穷项贝塞尔函数求和形式

经典的是求解 $\boldsymbol{j}_1(\boldsymbol{x},t)=\int f_1(\boldsymbol{x},\boldsymbol{v},t)\boldsymbol{v}d\boldsymbol{v}$

现在 $\boldsymbol{J}_1(\boldsymbol{x},t)=\int L_{\boldsymbol{G}}F\left(\boldsymbol{V}_c+\boldsymbol{w}\right)B_{0//}^{*}\delta\left(\left\|\boldsymbol{X}+\boldsymbol{\rho}-\boldsymbol{x}\right\|\right)d\boldsymbol{X}dud\mu d\theta$

$$\begin{aligned}\boldsymbol{J}_1(\boldsymbol{x},t)&=\int L_{\boldsymbol{G}}F\left(\boldsymbol{V}_c+\boldsymbol{w}\right)B_{0//}^{*}\delta\left(\left\|\boldsymbol{X}+\boldsymbol{\rho}-\boldsymbol{x}\right\|\right)d\boldsymbol{X}dud\mu d\theta\\&=B_{0//}^{*}\int dud\mu d\theta\delta\left(\left\|\boldsymbol{X}+\boldsymbol{\rho}-\boldsymbol{x}\right\|\right)\left(\boldsymbol{V}_c+\boldsymbol{w}\right)L_{\boldsymbol{G}}F\Big|_{\boldsymbol{X}->\boldsymbol{r}}\end{aligned}$$

其中的 $\delta\left(\left\|\boldsymbol{X}+\boldsymbol{\rho}-\boldsymbol{x}\right\|\right)=e^{-\boldsymbol{\rho}\cdot\nabla}=e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}=e^{-i\lambda\cos\theta}$ 项？

有 $\lambda=k_1\rho_0$，

利用 $\exp\left[-i\lambda\cos\theta\right]=\sum_{n'=-\infty}^{n'=\infty}I_{n'}\left(-i\lambda\right)e^{in'\theta}$

$$\begin{aligned}&\boldsymbol{J}_1(\boldsymbol{x},t)\\&=B_{0//}^{*}\int dud\mu d\theta e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}\left(u\boldsymbol{b}+\boldsymbol{w}\right)\{\left(-A_{1b}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}+i\boldsymbol{k}\cdot\boldsymbol{\rho}}-ik_bS\right)\partial_uF\\&+\left[-inS+e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}+i\boldsymbol{k}\cdot\boldsymbol{\rho}}\rho_0\left(A_{11}\sin\theta+A_{12}\cos\theta\right)\right]\partial_\mu F\}\end{aligned}$$

先计算 ∂_uF 在 $\boldsymbol{b}$ 方向上的值

其中利用如下几式：

$$\exp\left[i\lambda\cos\theta\right]=\sum_{n=-\infty}^{n=\infty}I_n\left(i\lambda\right)e^{in\theta}$$

$$\sum_{n=-\infty}^{n=\infty}J_n^2=1$$

$$I_n\left(i\lambda\right)I_{-n}\left(-i\lambda\right)=J_n^2\left(\lambda\right)$$

$$I_n'\left(i\lambda\right)I_{-n}\left(-i\lambda\right)=-iJ_n\left(\lambda\right)J_n'\left(\lambda\right)$$

将其中 $\boldsymbol{A}_1$ 的各个分量 A_{11},A_{12},A_{1b}，以及 ϕ_1 化简在一起。具体结果如下：

$$B_{0//}^{*}\int dud\mu d\theta e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}u\partial_{u}F\left(-A_{1b}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}+i\boldsymbol{k}\cdot\boldsymbol{\rho}}-ik_{b}S\right)$$

$$=-B_{0//}^{*}\int dud\mu d\theta u\partial_{u}F\sum_{n'=-\infty}^{n'=\infty}I_{n'}\left(-i\lambda\right)e^{in'\theta}\{A_{1b}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}+i\lambda\cos\theta}$$

$$+ik_{b}e^{\left(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}\right)}\sum_{n=-\infty}^{n=\infty}\frac{e^{in\theta}}{i\left(nB_{0}+k_{b}u-\omega\right)}[\left(\phi_{1}-A_{1b}u\right)I_{n}\left(i\lambda\right)+\sqrt{2\mu B_{0}}A_{11}\frac{nI_{n}\left(i\lambda\right)}{-\lambda}+\sqrt{2\mu B_{0}}A_{12}I_{n}^{'}\left(i\lambda\right)]$$

$$=-B_{0//}^{*}e^{\left(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}\right)}\int dud\mu d\theta\left(u\partial_{u}F\right)\sum_{n'=-\infty}^{n'=\infty}I_{n'}\left(-i\lambda\right)e^{in'\theta}\{\sum_{n=-\infty}^{n=\infty}I_{n}\left(i\lambda\right)e^{in\theta}A_{1b}$$

$$+ik_{b}\sum_{n=-\infty}^{n=\infty}\frac{e^{in\theta}}{i\left(nB_{0}+k_{b}u-\omega\right)}[\left(\phi_{1}-A_{1b}u\right)I_{n}\left(i\lambda\right)+\sqrt{2\mu B_{0}}A_{11}\frac{nI_{n}\left(i\lambda\right)}{-\lambda}+\sqrt{2\mu B_{0}}A_{12}I_{n}^{'}\left(i\lambda\right)]$$

$$=-B_{0//}^{*}e^{\left(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}\right)}\int dud\mu d\theta\left(u\partial_{u}F\right)\{\sum_{n=-\infty}^{n=\infty}J_{n}^{2}\left(\lambda\right)A_{1b}$$

$$+ik_{b}\sum_{n=-\infty}^{n=\infty}\frac{1}{i\left(nB_{0}+k_{b}u-\omega\right)}[\left(\phi_{1}-A_{1b}u\right)J_{n}^{2}\left(\lambda\right)$$

$$+\sqrt{2\mu B_{0}}\frac{nJ_{n}^{2}\left(\lambda\right)}{-\lambda}A_{11}-\sqrt{2\mu B_{0}}iJ_{n}\left(\lambda\right)J_{n}'\left(\lambda\right)A_{12}]$$

$$=-B_{0//}^{*}e^{\left(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}\right)}\int dud\mu d\theta\left(u\partial_{u}F\right)\sum_{n=-\infty}^{n=\infty}\{\frac{1}{\left(nB_{0}+k_{b}u-\omega\right)}[\left(nB_{0}-\omega\right)J_{n}^{2}\left(\lambda\right)A_{1b}+k_{b}J_{n}^{2}\left(\lambda\right)\phi_{1}$$

$$+\sqrt{2\mu B_{0}}k_{b}\frac{nJ_{n}^{2}\left(\lambda\right)}{-\lambda}A_{11}-\sqrt{2\mu B_{0}}ik_{b}J_{n}\left(\lambda\right)J_{n}'\left(\lambda\right)A_{12}]$$

积分项里面和θ相关的量已经约去，对θ积分即在前面乘以2π
该项和秦老师文章中对比，是完全一致的。

8、J₁ 的求解 2：有限项贝塞尔函数形式

$$
\begin{aligned}
&S(\boldsymbol{X},u,\mu,\theta,t)\\
&=e^{(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X})}\left[\left(\phi_1-uA_{1b}\right)g+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial g}{\partial\lambda}\right)\right]\\
&\nabla S(\boldsymbol{X},u,\mu,\theta,t)\\
&=i\boldsymbol{k}S\\
&\partial_\theta S(\boldsymbol{X},u,\mu,\theta,t)\\
&=e^{(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X})}\left[\left(\phi_1-uA_{1b}\right)\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial^2 g}{\partial\theta^2}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial^2 g}{\partial\theta\partial\lambda}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\boldsymbol{J}_1(\boldsymbol{x},t)\\
&=B_{0//}^*\int dud\mu d\theta e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}\left(u\boldsymbol{b}+\boldsymbol{w}\right)\{\left(-A_{1b}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}+i\boldsymbol{k}\cdot\boldsymbol{\rho}}-ik_bS\right)\partial_u F\\
&+\left[-\partial_\theta S+e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}+i\boldsymbol{k}\cdot\boldsymbol{\rho}}\rho_0\left(A_{11}\sin\theta+A_{12}\cos\theta\right)\right]\partial_\mu F\}\\
&=B_{0//}^*e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}\int dud\mu d\theta\left(u\boldsymbol{b}+\boldsymbol{w}\right)\\
&\{\left(-A_{1b}-ik_be^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}\left[\left(\phi_1-uA_{1b}\right)g+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial g}{\partial\lambda}\right)\right]\right)\partial_u F\\
&+\left[-e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}\left[\left(\phi_1-uA_{1b}\right)\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial^2 g}{\partial\theta^2}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial^2 g}{\partial\theta\partial\lambda}\right)\right]+\rho_0\left(A_{11}\sin\theta+A_{12}\cos\theta\right)\right]\partial_\mu F\}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{J}_1(\boldsymbol{x},t)=\begin{pmatrix}J_{11}\\J_{12}\\J_{1b}\end{pmatrix}&=\begin{pmatrix}\chi_{11}&\chi_{12}&\chi_{1b}&\chi_1\\\chi_{21}&\chi_{22}&\chi_{2b}&\chi_2\\\chi_{31}&\chi_{32}&\chi_{3b}&\chi_3\end{pmatrix}\begin{pmatrix}A_{11}\\A_{12}\\A_{1b}\\\phi_1\end{pmatrix}\\
&=B_{0//}^*e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}\int dud\mu\partial_u F\begin{pmatrix}a_{11}&a_{12}&a_{1b}&a_1\\a_{21}&a_{22}&a_{2b}&a_2\\a_{31}&a_{32}&a_{3b}&a_3\end{pmatrix}\begin{pmatrix}A_{11}\\A_{12}\\A_{1b}\\\phi_1\end{pmatrix}\\
&+B_{0//}^*e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}\int dud\mu\partial_\mu F\begin{pmatrix}b_{11}&b_{12}&b_{1b}&b_1\\b_{21}&b_{22}&b_{2b}&b_2\\b_{31}&b_{32}&b_{3b}&b_3\end{pmatrix}\begin{pmatrix}A_{11}\\A_{12}\\A_{1b}\\\phi_1\end{pmatrix}
\end{aligned}
$$

以下逐项计算各个分量：

8.1 J_{1b}

8.1.1 J_{1b} 中 $\partial_u F$ 的量

$$
\begin{aligned}
&B_{0//}^{*}\int dud\mu d\theta e^{-ik\cdot\rho}u\partial_u F\left(-A_{1b}e^{-i\omega t+ik\cdot X+ik\cdot\rho}-ik_b S\right)\\
&=-B_{0//}^{*}\int dud\mu d\theta\left(u\partial_u F\right)\left(A_{1b}e^{-i\omega t+ik\cdot X}+ik_b Se^{-i\lambda\cos\theta}\right)\\
&=-B_{0//}^{*}e^{-i\omega t+ik\cdot X}\int dud\mu d\theta\left(u\partial_u F\right)\{A_{1b}\\
&+ik_b e^{-i\lambda\cos\theta}\left[\left(\phi_1-uA_{1b}\right)g+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial g}{\partial\lambda}\right)\right]\}
\end{aligned}
$$

要求解的项有：$\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}g$，$\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta}$，$\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\lambda}$

前面的 2π 是必须的？

第一项

$$
\begin{aligned}
&\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}g\\
&=\frac{1}{2\pi}\int_0^{2\pi}d\theta e^{-i\lambda\cos\theta}\frac{C_0}{2\pi}\int_0^{2\pi}\exp\left[-iat+i\lambda\cos\left(\theta+t\right)\right]dt\\
&=\frac{C_0}{2\pi}\int_0^{2\pi}d\theta\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-iat+i\lambda\cos\left(\theta+t\right)-i\lambda\cos\theta\right]dt\\
&=\frac{C_0}{2\pi}\int_0^{2\pi}\exp\left(-iat\right)dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[i\lambda\cos\left(\theta+t\right)-i\lambda\cos\theta\right]d\theta\\
&=\frac{C_0}{2\pi}\int_0^{2\pi}\exp\left(-iat\right)dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\left(\theta+\frac{t}{2}\right)\sin\left(\frac{t}{2}\right)\right]d\theta
\end{aligned}
$$

$$
\begin{aligned}
&J_0\left(x\right)=\frac{1}{2\pi}\int_0^{2\pi}d\alpha\exp\left[ix\sin\alpha\right]\\
&J_0\left(x\right)=\frac{1}{2\pi}\int_0^{2\pi}d\left(\theta+\frac{t}{2}\right)\exp\left[ix\sin\left(\theta+\frac{t}{2}\right)\right]\\
&=\frac{1}{2\pi}\int_{0+\frac{t}{2}}^{2\pi+\frac{t}{2}}\exp\left[-2i\lambda\sin\left(\theta+\frac{t}{2}\right)\sin\left(\frac{t}{2}\right)\right]d\theta\\
&=J_0\left(-2\lambda\sin\frac{t}{2}\right)
\end{aligned}
$$

则有 $x=-2\lambda\sin\left(\frac{t}{2}\right)$，积分限的变化对贝塞尔函数 J 的影响？

代入上式得：表示为 G_0

$$G_0=\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}g$$
$$=\frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)\,dt\,\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\left(\theta+\frac{t}{2}\right)\sin\left(\frac{t}{2}\right)\right]d\theta$$
$$=\frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)\,dt\,J_0\left(-2\lambda\sin\frac{t}{2}\right)$$

利用 $J_{-m}(\lambda)J_m(\lambda)=\frac{1}{2\pi}e^{(im\pi)}\int_0^{2\pi}d\beta J_0\left(2\lambda\sin\frac{\beta}{2}\right)e^{-im\beta}$

即 $J_{-a}(-\lambda)J_a(-\lambda)=\frac{1}{2\pi}e^{(ia\pi)}\int_0^{2\pi}dtJ_0\left(-2\lambda\sin\frac{t}{2}\right)e^{-iat}$

$$G_0=\frac{C_0}{2\pi}\int_0^{2\pi}dt\exp(-iat)J_0\left(-2\lambda\sin\frac{t}{2}\right)=C_0e^{(-ia\pi)}J_{-a}(-\lambda)J_a(-\lambda)$$

第二项

$$\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta}=\frac{1}{2\pi}e^{-i\lambda\cos\theta}g\Big|_0^{2\pi}-\frac{1}{2\pi}\int d\theta(i\lambda\sin\theta)e^{-i\lambda\cos\theta}g$$
$$=-i\lambda\frac{1}{2\pi}\int d\theta(\sin\theta)e^{-i\lambda\cos\theta}g$$
$$=-i\lambda G_1$$

其中

$$G_1 = \frac{1}{2\pi}\int d\theta(\sin\theta)e^{-i\lambda\cos\theta}g$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta(\sin\theta)e^{-i\lambda\cos\theta}\frac{C_0}{2\pi}\int_0^{2\pi}\exp\left[-iat+i\lambda\cos(\theta+t)\right]dt$$

$$= \frac{C_0}{2\pi}\int_0^{2\pi} d\theta\frac{1}{2\pi}\int_0^{2\pi}(\sin\theta)\exp\left[-iat+i\lambda\cos(\theta+t)-i\lambda\cos\theta\right]dt$$

$$= \frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)\,dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[i\lambda\cos(\theta+t)-i\lambda\cos\theta\right](\sin\theta)\,d\theta$$

$$= \frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)\,dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\left(\theta+\frac{t}{2}\right)\sin\left(\frac{t}{2}\right)\right](\sin\theta)\,d\theta$$

$$= \frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)\,dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right]\sin\left(\phi-\frac{t}{2}\right)d\phi$$

$$= \frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)\,dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right]\left[\sin\phi\cos\frac{t}{2}-\cos\phi\sin\frac{t}{2}\right]d\phi$$

$$= \frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)\,dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right]\sin\phi\cos\frac{t}{2}\,d\phi$$

$$= \frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)\,dt\frac{\partial}{\partial t}\frac{1}{-i\lambda}\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right]d\phi$$

$$= \frac{1}{-i\lambda}\frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)\,dt\frac{\partial}{\partial t}J_0\left(-2\lambda\sin\frac{t}{2}\right)$$

$$= \frac{1}{-i\lambda}\left[\frac{C_0}{2\pi}\exp(-iat)J_0\left(-2\lambda\sin\frac{t}{2}\right)\Bigg|_0^{2\pi}-\frac{C_0}{2\pi}\int_0^{2\pi}\frac{\partial e^{-iat}}{\partial t}J_0\left(-2\lambda\sin\frac{t}{2}\right)dt\right]$$

$$= \frac{1}{-i\lambda}\left[\frac{C_0}{2\pi}\exp(-ia2\pi)-\frac{C_0}{2\pi}+ia\frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)J_0\left(-2\lambda\sin\frac{t}{2}\right)dt\right]$$

$$= \frac{1}{-i\lambda}\left[\frac{C_0}{2\pi}\left[\exp(-ia2\pi)-1\right]+ia\frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)J_0\left(-2\lambda\sin\frac{t}{2}\right)dt\right]$$

$$= \frac{1}{-i\lambda}\frac{C_0}{2\pi}\left[\exp(-ia2\pi)-1\right]+\frac{ia}{-i\lambda}G_0$$

即第二项为：

$$\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta}$$

$$= \frac{C_0}{2\pi}\left[\exp(-ia2\pi)-1\right]+iaG_0$$

第三项

$$\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\lambda}=\frac{\partial}{\partial\lambda}\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}g+\frac{1}{2\pi}\int d\theta i\cos\theta e^{-i\lambda\cos\theta}g$$

$$=\frac{\partial}{\partial\lambda}G_0+i\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}g$$

其中

$$G_2=\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}g$$

$$=\frac{1}{2\pi}\int_0^{2\pi}d\theta(\cos\theta)e^{-i\lambda\cos\theta}\frac{C_0}{2\pi}\int_0^{2\pi}\exp\left[-iat+i\lambda\cos(\theta+t)\right]dt$$

$$=\frac{C_0}{2\pi}\int_0^{2\pi}d\theta\frac{1}{2\pi}\int_0^{2\pi}(\cos\theta)\exp\left[-iat+i\lambda\cos(\theta+t)-i\lambda\cos\theta\right]dt$$

$$=\frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[i\lambda\cos(\theta+t)-i\lambda\cos\theta\right](\cos\theta)d\theta$$

$$=\frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\left(\theta+\frac{t}{2}\right)\sin\left(\frac{t}{2}\right)\right](\cos\theta)d\theta$$

$$=\frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right]\cos\left(\phi-\frac{t}{2}\right)d\phi$$

$$=\frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right]\left[\cos\phi\cos\frac{t}{2}+\sin\phi\sin\frac{t}{2}\right]d\phi$$

$$=\frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right]\sin\phi\sin\frac{t}{2}d\phi$$

$$=\frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)dt\frac{\partial}{\partial\lambda}\frac{1}{-2i}\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right]d\phi$$

$$=\frac{1}{-2i}\frac{\partial}{\partial\lambda}\frac{C_0}{2\pi}\int_0^{2\pi}\exp(-iat)dtJ_0\left(-2\lambda\sin\frac{t}{2}\right)$$

$$=\frac{1}{-2i}\frac{\partial}{\partial\lambda}G_0$$

代入上式得，第三项为：

$$\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\lambda}$$

$$=\frac{\partial}{\partial\lambda}G_0+i\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}g$$

$$=\frac{\partial}{\partial\lambda}G_0+i\frac{1}{-2i}\frac{\partial}{\partial\lambda}G_0$$

$$=\frac{1}{2}\frac{\partial}{\partial\lambda}G_0$$

代入得 J1b 中 $\partial_u F$ 的量为

$$B_{0//}^{*}\int dud\mu d\theta e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}u\partial_u F\left(-A_{1b}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}+i\boldsymbol{k}\cdot\boldsymbol{\rho}}-ik_b S\right)$$
$$=-B_{0//}^{*}\int dud\mu d\theta\left(u\partial_u F\right)\left(A_{1b}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}+ik_b Se^{-i\lambda\cos\theta}\right)$$
$$=-B_{0//}^{*}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}\int dud\mu d\theta\left(u\partial_u F\right)\{A_{1b}$$
$$+ik_b e^{-i\lambda\cos\theta}\left[\left(\phi_1-uA_{1b}\right)g+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial g}{\partial\lambda}\right)\right]\}$$
$$=-B_{0//}^{*}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}\int dud\mu\left(u\partial_u F\right)\{A_{1b}$$
$$+ik_b\left(\phi_1-uA_{1b}\right)G_0+ik_b\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\left[\frac{C_0}{2\pi}\left(e^{-ia2\pi}-1\right)+iaG_0\right]+ik_b\sqrt{2\mu B_0}A_{12}\left(-\frac{i}{2}\frac{\partial}{\partial\lambda}G_0\right)\}$$
$$=B_{0//}^{*}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}\int dud\mu\left(-u\partial_u F\right)\{-k_b\frac{B_0}{k_1}\left[\frac{C_0}{2\pi}\left(e^{-ia2\pi}-1\right)+iaG_0\right]A_{11}$$
$$+k_b\sqrt{2\mu B_0}\left(\frac{1}{2}\frac{\partial}{\partial\lambda}G_0\right)A_{12}+\left(1-ik_b uG_0\right)A_{1b}+ik_b G_0\phi_1\}$$

8.1.2 J1b 中 $\partial_\mu F$ 的量

$$B_{0//}^{*}\int dud\mu d\theta e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}\left(u\right)\left[-\partial_\theta S+e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}+i\boldsymbol{k}\cdot\boldsymbol{\rho}}\rho_0\left(A_{11}\sin\theta+A_{12}\cos\theta\right)\right]\partial_\mu F$$
$$=B_{0//}^{*}\int dud\mu d\theta\left(u\right)\left[-\partial_\theta Se^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}+e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}\rho_0\left(A_{11}\sin\theta+A_{12}\cos\theta\right)\right]\partial_\mu F$$
$$=B_{0//}^{*}\int dud\mu d\theta\left(u\right)[-e^{\left(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}\right)}e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}\left[\left(\phi_1-uA_{1b}\right)\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial^2 g}{\partial\theta^2}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial^2 g}{\partial\theta\partial\lambda}\right)\right]$$
$$+e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}\rho_0\left(A_{11}\sin\theta+A_{12}\cos\theta\right)]\partial_\mu F$$
$$=B_{0//}^{*}e^{\left(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}\right)}\int dud\mu d\theta\left(u\partial_\mu F\right)\{-e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}\left[\left(\phi_1-uA_{1b}\right)\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial^2 g}{\partial\theta^2}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial^2 g}{\partial\theta\partial\lambda}\right)\right]$$
$$+\rho_0\left(A_{11}\sin\theta+A_{12}\cos\theta\right)\}$$
$$=-B_{0//}^{*}e^{\left(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}\right)}\int dud\mu d\theta\left(u\partial_\mu F\right)\left(e^{-i\lambda\cos\theta}\right)\left[\left(\phi_1-uA_{1b}\right)\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial^2 g}{\partial\theta^2}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial^2 g}{\partial\theta\partial\lambda}\right)\right]$$

需要求解的积分项有：$\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta}$，$\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}\frac{\partial^2 g}{\partial\theta^2}$，$\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}\frac{\partial^2 g}{\partial\theta\partial\lambda}$

第一项已求，

$$\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta}$$
$$=\frac{C_0}{2\pi}\left[\exp(-ia2\pi)-1\right]+iaG_0$$

第二项:

$$\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}\frac{\partial^2 g}{\partial\theta^2}$$
$$=\frac{1}{2\pi}e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta}\bigg|_0^{2\pi}-\frac{1}{2\pi}\int d\theta(-i\lambda\sin\theta)e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta}$$
$$=i\lambda\frac{1}{2\pi}\int d\theta(\sin\theta)e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta}$$
$$=i\lambda\frac{1}{2\pi}\int d\theta(\sin\theta)e^{-i\lambda\cos\theta}\frac{C_0}{2\pi}\int_0^{2\pi}\left[-i\lambda\sin(\theta+t)\right]\exp\left[-iat+i\lambda\cos(\theta+t)\right]dt$$
$$=\lambda^2C_0\frac{1}{2\pi}\int e^{-iat}dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[i\lambda\cos(\theta+t)-i\lambda\cos\theta\right]\sin(\theta+t)\sin\theta d\theta$$
$$=\lambda^2C_0\frac{1}{2\pi}\int e^{-iat}dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\left(\theta+\frac{t}{2}\right)\sin\left(\frac{t}{2}\right)\right]\sin(\theta+t)\sin\theta d\theta$$
$$=\lambda^2C_0\frac{1}{2\pi}\int e^{-iat}dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right]\sin\left(\phi+\frac{t}{2}\right)\sin\left(\phi-\frac{t}{2}\right)d\phi$$
$$=\lambda^2C_0\frac{1}{2\pi}\int e^{-iat}dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right]\left[-\frac{1}{2}(\cos2\phi-\cos t)\right]d\phi$$
$$=\lambda^2C_0\frac{1}{2\pi}\int e^{-iat}dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right]\left[-\frac{1}{2}(1-\cos t-2\sin^2\phi)\right]d\phi$$

其中前一项

$$\lambda^2C_0\frac{1}{2\pi}\int\left[-\frac{1}{2}(1-\cos t)\right]e^{-iat}dtJ_0\left(-2\lambda\sin\frac{t}{2}\right)$$
$$=-\frac{1}{2}\lambda^2\left[C_0\frac{1}{2\pi}\int e^{-iat}dtJ_0\left(-2\lambda\sin\frac{t}{2}\right)+C_0\frac{1}{2\pi}\int(-\cos t)e^{-iat}dtJ_0\left(-2\lambda\sin\frac{t}{2}\right)\right]$$
$$=-\frac{1}{2}\lambda^2G_0$$

后一项

$$\lambda^2 C_0 \frac{1}{2\pi}\int e^{-iat}dt \frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right]\left(\sin^2\phi\right)d\phi$$

$$=-\lambda^2 C_0 \frac{1}{2\pi}\int e^{-iat}dtJ_0''\left(-2\lambda\sin\frac{t}{2}\right)$$

$$=-\lambda^2 C_0 \frac{1}{2\pi}\int e^{-iat}dt\left(\frac{1}{4}\frac{\partial^2 J_0}{\partial\lambda^2}+\frac{1}{\lambda^2}\frac{\partial^2 J_0}{\partial t^2}+\frac{1}{4\lambda}\frac{\partial J_0}{\partial\lambda}\right)$$

$$=-\lambda^2\left(\frac{1}{4}\frac{\partial^2}{\partial\lambda^2}+\frac{1}{4\lambda}\frac{\partial}{\partial\lambda}\right)G_0-\lambda^2 C_0\frac{1}{2\pi}\int e^{-iat}dt\left(\frac{1}{\lambda^2}\frac{\partial^2 J_0}{\partial t^2}\right)$$

$$=-\lambda^2\left(\frac{1}{4}\frac{\partial^2}{\partial\lambda^2}+\frac{1}{4\lambda}\frac{\partial}{\partial\lambda}\right)G_0-(ia)^2 C_0\frac{1}{2\pi}\int e^{-iat}dtJ_0$$

$$=\left(-\frac{\lambda^2}{4}\frac{\partial^2}{\partial\lambda^2}-\frac{\lambda}{4}\frac{\partial}{\partial\lambda}+a^2\right)G_0$$

相加得

$$\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}\frac{\partial^2 g}{\partial\theta^2}$$

$$=\left(-\frac{1}{2}\lambda^2-\frac{\lambda^2}{4}\frac{\partial^2}{\partial\lambda^2}-\frac{\lambda}{4}\frac{\partial}{\partial\lambda}+a^2\right)G_0$$

其中由参考文献【2】中 A12,A13 得

$$J_0''=\frac{1}{4}\frac{\partial^2 J_0}{\partial\lambda^2}+\frac{1}{\lambda^2}\frac{\partial^2 J_0}{\partial t^2}+\frac{1}{4\lambda}\frac{\partial J_0}{\partial\lambda}$$

第三项

$$\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}\frac{\partial^2 g}{\partial\theta\partial\lambda}$$

$$=\frac{1}{2\pi}e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\lambda}\bigg|_0^{2\pi}-\frac{1}{2\pi}\int d\theta\frac{\partial e^{-i\lambda\cos\theta}}{\partial\theta}\frac{\partial g}{\partial\lambda}$$

$$=-\frac{1}{2\pi}\int d\theta\left(i\lambda\sin\theta\right)e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\lambda}$$

$$=-\frac{i\lambda}{2\pi}\int d\theta\left(\sin\theta\right)e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\lambda}$$

其中

$$\frac{1}{2\pi}\int d\theta(\sin\theta)e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\lambda}$$
$$=\frac{1}{2\pi}\int d\theta(\sin\theta)e^{-i\lambda\cos\theta}\frac{C_0}{2\pi}\int_0^{2\pi}i\cos(\theta+t)\exp\left[-iat+i\lambda\cos(\theta+t)\right]dt$$
$$=iC_0\frac{1}{2\pi}\int e^{-iat}dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[i\lambda\cos(\theta+t)-i\lambda\cos\theta\right]\sin\theta\cos(\theta+t)d\theta$$
$$=iC_0\frac{1}{2\pi}\int e^{-iat}dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\left(\theta+\frac{t}{2}\right)\sin\left(\frac{t}{2}\right)\right]\sin\theta\cos(\theta+t)d\theta$$
$$=iC_0\frac{1}{2\pi}\int e^{-iat}dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right]\sin\left(\phi-\frac{t}{2}\right)\cos\left(\phi+\frac{t}{2}\right)d\theta$$
$$=iC_0\frac{1}{2\pi}\int e^{-iat}dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right](\sin 2\phi-\sin t)d\theta$$
$$=iC_0\frac{1}{2\pi}\int e^{-iat}dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right](2\sin\phi\cos\phi)d\theta$$
$$+iC_0\frac{1}{2\pi}\int e^{-iat}dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right](-\sin t)d\theta$$
$$=iC_0\frac{1}{2\pi}\int e^{-iat}dt\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-2i\lambda\sin\phi\sin\left(\frac{t}{2}\right)\right](2\sin\phi)d(\sin\phi)$$
$$+iC_0\frac{1}{2\pi}\int e^{-iat}(-\sin t)dtJ_0\left(-2\lambda\sin\frac{t}{2}\right)$$
$$=iC_0\frac{1}{2\pi}\int e^{-iat}\left(-2\sin\frac{t}{2}\cos\frac{t}{2}\right)J_0\left(-2\lambda\sin\frac{t}{2}\right)dt$$
$$=iC_0\frac{1}{2\pi}\int e^{-iat}\left(-\frac{1}{\lambda}\right)\frac{\partial^2 J_0}{\partial\lambda\partial t}dt$$
$$=i\left(-\frac{1}{\lambda}\right)C_0\frac{1}{2\pi}\int iae^{-iat}\frac{\partial J_0}{\partial\lambda}dt$$
$$=\frac{a}{\lambda}\frac{\partial}{\partial\lambda}G_0$$

则第三项

$$\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}\frac{\partial^2 g}{\partial\theta\partial\lambda}$$
$$=-\frac{i\lambda}{2\pi}\int d\theta(\sin\theta)e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\lambda}$$
$$=-i\lambda\frac{a}{\lambda}\frac{\partial}{\partial\lambda}G_0$$
$$=-ia\frac{\partial}{\partial\lambda}G_0$$

代入得 **J1b** 中 $\partial_\mu F$ 的量

$$B_{0//}^{*}\int dud\mu d\theta e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}\left(u\right)\left[-\partial_\theta S+e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}+i\boldsymbol{k}\cdot\boldsymbol{\rho}}\rho_0\left(A_{11}\sin\theta+A_{12}\cos\theta\right)\right]\partial_\mu F$$

$$=-B_{0//}^{*}e^{\left(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}\right)}\int dud\mu d\theta\left(u\partial_\mu F\right)\left(e^{-i\lambda\cos\theta}\right)\left[\left(\phi_1-uA_{1b}\right)\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial^2 g}{\partial\theta^2}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial^2 g}{\partial\theta\partial\lambda}\right)\right]$$

$$=-B_{0//}^{*}e^{\left(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}\right)}\int dud\mu d\theta\left(u\partial_\mu F\right)\{\left(\phi_1-uA_{1b}\right)\left[\frac{C_0}{2\pi}\left(e^{-ia2\pi}-1\right)+iaG_0\right]$$

$$+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\left(-\frac{1}{2}\lambda^2-\frac{\lambda^2}{4}\frac{\partial^2}{\partial\lambda^2}-\frac{\lambda}{4}\frac{\partial}{\partial\lambda}+a^2\right)G_0-i\sqrt{2\mu B_0}A_{12}\left(-ia\frac{\partial}{\partial\lambda}G_0\right)\}$$

$$=B_{0//}^{*}e^{\left(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}\right)}\int dud\mu\left(-u\partial_\mu F\right)\{i\left(-\frac{1}{2}\lambda-\frac{\lambda}{4}\frac{\partial^2}{\partial\lambda^2}-\frac{1}{4}\frac{\partial}{\partial\lambda}+a^2\right)G_0\sqrt{2\mu B_0}A_{11}$$

$$-\sqrt{2\mu B_0}\left(a\frac{\partial}{\partial\lambda}G_0\right)A_{12}+\left[\frac{C_0}{2\pi}\left(e^{-ia2\pi}-1\right)+iaG_0\right]\left(\phi_1-uA_{1b}\right)\}$$

8.2 J_{11}

8.2.1 J11 中 $\partial_u F$ 的量

$$\rho_0=\sqrt{\frac{2\mu}{B_0}}\text{，}\quad \lambda=k_1\sqrt{\frac{2\mu}{B_0}}=k_1\rho_0\text{，}$$

$$\boldsymbol{\rho}=\sqrt{\frac{2\mu}{B_0}}\boldsymbol{a}=\sqrt{\frac{2\mu}{B_0}}(\boldsymbol{e}_1\cos\theta-\boldsymbol{e}_2\sin\theta)=\rho_0(\boldsymbol{e}_1\cos\theta-\boldsymbol{e}_2\sin\theta)$$

$$\boldsymbol{w}=\sqrt{2\mu B_0}\boldsymbol{c}=\sqrt{2\mu B_0}(-\boldsymbol{e}_1\sin\theta-\boldsymbol{e}_2\cos\theta)=\rho_0 B_0(-\boldsymbol{e}_1\sin\theta-\boldsymbol{e}_2\cos\theta)$$

$$=\frac{\lambda}{k_1}B_0(-\boldsymbol{e}_1\sin\theta-\boldsymbol{e}_2\cos\theta)$$

$$J_{11}(\boldsymbol{x},t)$$

$$=B_{0//}^{*}\int dud\mu d\theta e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}\left(-\sqrt{2\mu B_0}\sin\theta\right)\left(-A_{1b}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}+i\boldsymbol{k}\cdot\boldsymbol{\rho}}-ik_bS\right)\partial_u F$$

$$=B_{0//}^{*}\int dud\mu d\theta\left(-\sqrt{2\mu B_0}\sin\theta\right)\left(-A_{1b}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}-ik_bSe^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}\right)\partial_u F$$

$$=B_{0//}^{*}\int dud\mu d\theta\left(\sqrt{2\mu B_0}\sin\theta\right)\partial_u F\{A_{1b}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}$$

$$+ik_be^{-i\lambda\cos\theta}e^{(-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X})}\left[\left(\phi_1-\boldsymbol{A}_1\cdot\boldsymbol{V}_c\right)g+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial g}{\partial\lambda}\right)\right]\}$$

$$=B_{0//}^{*}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}\int dud\mu d\theta\left(\sqrt{2\mu B_0}\sin\theta\right)\partial_u F\{A_{1b}$$

$$+ik_be^{-i\lambda\cos\theta}\left[\left(\phi_1-\boldsymbol{A}_1\cdot\boldsymbol{V}_c\right)g+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial g}{\partial\lambda}\right)\right]\}$$

要求解的积分项有：

$$\frac{1}{2\pi}\int d\theta\sin\theta e^{-i\lambda\cos\theta}g\text{，}\quad\frac{1}{2\pi}\int d\theta\sin\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta}\text{，}\quad\frac{1}{2\pi}\int d\theta\sin\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\lambda}$$

这三项在前面都已经求出：如下：

$$\frac{1}{2\pi}\int d\theta\sin\theta e^{-i\lambda\cos\theta}g=G_1$$

$$=\frac{1}{-i\lambda}\frac{C_0}{2\pi}\left[\exp\left(-ia2\pi\right)-1\right]+\frac{ia}{-i\lambda}G_0$$

$$=-\frac{1}{i\lambda}\left[\frac{C_0}{2\pi}\left(e^{-ia2\pi}-1\right)+iaG_0\right]$$

$$\frac{1}{2\pi}\int d\theta \sin\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta}=G_3$$
$$=-i\left(-\frac{1}{2}\lambda-\frac{\lambda}{4}\frac{\partial^2}{\partial\lambda^2}-\frac{1}{4}\frac{\partial}{\partial\lambda}+a^2\right)G_0$$

$$\frac{1}{2\pi}\int d\theta \sin\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\lambda}=G_4$$
$$=\frac{a}{\lambda}\frac{\partial}{\partial\lambda}G_0$$

代入得 J11 中 $\partial_u F$ 的量为

$$B_{0//}^{*}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}\int dud\mu d\theta\left(\sqrt{2\mu B_0}\sin\theta\right)\partial_u F\{A_{1b}$$
$$+ik_b e^{-i\lambda\cos\theta}\left[\left(\phi_1-\boldsymbol{A}_1\cdot\boldsymbol{V}_c\right)g+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial g}{\partial\lambda}\right)\right]\}$$
$$=B_{0//}^{*}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}\int dud\mu\sqrt{2\mu B_0}\partial_u F$$
$$ik_b\{\left[\left(\phi_1-A_{11}u\right)G_1+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}G_3+\sqrt{2\mu B_0}A_{12}\left(-iG_4\right)\right]\}$$
$$=B_{0//}^{*}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}\int dud\mu\partial_u F\sqrt{2\mu B_0}$$
$$ik_b\{\left[\left(\frac{i}{\lambda}\sqrt{2\mu B_0}G_3-uG_1\right)A_{11}-iG_4\sqrt{2\mu B_0}A_{12}+G_1\phi_1\right]\}$$

8.2.1 J11 中 $\partial_\mu F$ 的量

需要求解的积分项有：

$$\frac{1}{2\pi}\int d\theta \sin\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta},\quad \frac{1}{2\pi}\int d\theta \sin\theta e^{-i\lambda\cos\theta}\frac{\partial^2 g}{\partial\theta^2},\quad \frac{1}{2\pi}\int d\theta \sin\theta e^{-i\lambda\cos\theta}\frac{\partial^2 g}{\partial\theta\partial\lambda}$$

具体如下：

第一项：

$$\frac{1}{2\pi}\int d\theta \sin\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta}=G_3$$
$$=-i\left(-\frac{1}{2}\lambda-\frac{\lambda}{4}\frac{\partial^2}{\partial\lambda^2}-\frac{1}{4}\frac{\partial}{\partial\lambda}+a^2\right)G_0$$

第二项：

$$\frac{1}{2\pi}\int d\theta \sin\theta e^{-i\lambda\cos\theta}\frac{\partial^2 g}{\partial\theta^2}=G_5$$

第三项：

$$\frac{1}{2\pi}\int d\theta \sin\theta e^{-i\lambda\cos\theta}\frac{\partial^2 g}{\partial\theta\partial\lambda}=G_6$$

代入得 **J11** 中 $\partial_\mu F$ 的量为

$$B_{0//}^{*}e^{-i\omega t+ik\cdot X}\int dud\mu d\theta\left(-\sqrt{2\mu B_0}\sin\theta\right)\partial_\mu F$$
$$\{-e^{-ik\cdot\rho}\left[\left(\phi_1-uA_{1b}\right)\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial^2 g}{\partial\theta^2}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial^2 g}{\partial\theta\partial\lambda}\right)\right]+\rho_0\left(A_{11}\sin\theta+A_{12}\cos\theta\right)\}$$
$$=B_{0//}^{*}e^{-i\omega t+ik\cdot X}\int dud\mu d\theta\left(\sqrt{2\mu B_0}\right)\sin\theta\partial_\mu F$$
$$\{e^{-i\lambda\cos\theta}\left[\left(\phi_1-uA_{1b}\right)\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial^2 g}{\partial\theta^2}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial^2 g}{\partial\theta\partial\lambda}\right)\right]-\rho_0\left(A_{11}\sin\theta+A_{12}\cos\theta\right)\}$$
$$=B_{0//}^{*}e^{-i\omega t+ik\cdot X}\int dud\mu\left(\sqrt{2\mu B_0}\right)\partial_\mu F$$
$$\{\left[\left(\phi_1-uA_{1b}\right)G_3+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}G_5+\sqrt{2\mu B_0}A_{12}\left(-iG_6\right)\right]-\frac{\rho_0}{2}A_{11}\}$$
$$=B_{0//}^{*}e^{-i\omega t+ik\cdot X}\int dud\mu\partial_\mu F\left(\sqrt{2\mu B_0}\right)$$
$$\{\left(\frac{i}{\lambda}\sqrt{2\mu B_0}G_5-\frac{\rho_0}{2}\right)A_{11}-i\sqrt{2\mu B_0}G_6A_{12}-uG_3A_{1b}+G_3\phi_1\}$$

8.3 J_{12}

与上面 J_{11} 类似，只是将$\sin\theta$都换成$\cos\theta$

8.3.1 J12 中$\partial_u F$ 的量

$$
\begin{aligned}
& J_{11}(\boldsymbol{x},t) \\
& = B_{0//}^{*} e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{X}} \int du d\mu d\theta \left(\sqrt{2\mu B_0}\cos\theta\right) \partial_u F \{ A_{1b} \\
& + i k_b e^{-i\lambda\cos\theta} \left[\left(\phi_1 - \boldsymbol{A}_1 \cdot \boldsymbol{V}_c \right) g + \sqrt{2\mu B_0} A_{11} \frac{i}{\lambda} \frac{\partial g}{\partial\theta} + \sqrt{2\mu B_0} A_{12} \left(-i \frac{\partial g}{\partial\lambda} \right) \right] \}
\end{aligned}
$$

要求解的积分项有：

$\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta} g$，$\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta}$，$\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\lambda}$

具体如下：

$$
\begin{aligned}
& \frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta} g = G_2 \\
& = \frac{1}{-2i}\frac{\partial}{\partial\lambda} G_0
\end{aligned}
$$

$$
\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta} = G_7
$$

$$
\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\lambda} = G_8
$$

代入得 J_{12} 中 $\partial_u F$ 的量为

$$
\begin{aligned}
& B_{0//}^{*} e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}} \int dud\mu d\theta \left(\sqrt{2\mu B_0}\cos\theta\right)\partial_u F\{A_{1b} \\
& +ik_b e^{-i\lambda\cos\theta}\left[\left(\phi_1 - \boldsymbol{A}_1\cdot\boldsymbol{V}_c\right)g+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial g}{\partial\lambda}\right)\right]\} \\
& = B_{0//}^{*} e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}} \int dud\mu\sqrt{2\mu B_0}\partial_u F \\
& ik_b\{\left[\left(\phi_1 - A_{11}u\right)G_2+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}G_7+\sqrt{2\mu B_0}A_{12}\left(-iG_8\right)\right]\} \\
& = B_{0//}^{*} e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}} \int dud\mu\partial_u F\sqrt{2\mu B_0} \\
& ik_b\{\left[\left(\frac{i}{\lambda}\sqrt{2\mu B_0}G_7 - uG_1\right)A_{11} - iG_8\sqrt{2\mu B_0}A_{12}+G_2\phi_1\right]\}
\end{aligned}
$$

8.2.1 J_{12} 中 $\partial_\mu F$ 的量

需要求解的积分项有：

$$\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta}，\quad \frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}\frac{\partial^2 g}{\partial\theta^2}，\quad \frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}\frac{\partial^2 g}{\partial\theta\partial\lambda}$$

具体如下：

第一项：

$$\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta}=G_7$$

第二项：

$$\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}\frac{\partial^2 g}{\partial\theta^2}=G_9$$

第三项：

$$\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}\frac{\partial^2 g}{\partial\theta\partial\lambda}=G_{10}$$

代入得 J12 中 $\partial_\mu F$ 的量为

$$B_{0//}^{*}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}\int dud\mu d\theta\left(-\sqrt{2\mu B_0}\cos\theta\right)\partial_\mu F$$

$$\{-e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}\left[\left(\phi_1-uA_{1b}\right)\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial^2 g}{\partial\theta^2}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial^2 g}{\partial\theta\partial\lambda}\right)\right]+\rho_0\left(A_{11}\sin\theta+A_{12}\cos\theta\right)\}$$

$$=B_{0//}^{*}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}\int dud\mu d\theta\left(\sqrt{2\mu B_0}\right)\cos\theta\partial_\mu F$$

$$\{e^{-i\lambda\cos\theta}\left[\left(\phi_1-uA_{1b}\right)\frac{\partial g}{\partial\theta}+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}\frac{\partial^2 g}{\partial\theta^2}+\sqrt{2\mu B_0}A_{12}\left(-i\frac{\partial^2 g}{\partial\theta\partial\lambda}\right)\right]-\rho_0\left(A_{11}\sin\theta+A_{12}\cos\theta\right)\}$$

$$=B_{0//}^{*}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}\int dud\mu\left(\sqrt{2\mu B_0}\right)\partial_\mu F$$

$$\{\left[\left(\phi_1-uA_{1b}\right)G_7+\sqrt{2\mu B_0}A_{11}\frac{i}{\lambda}G_9+\sqrt{2\mu B_0}A_{12}\left(-iG_{10}\right)\right]-\frac{\rho_0}{2}A_{12}\}$$

$$=B_{0//}^{*}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{X}}\int dud\mu\partial_\mu F\sqrt{2\mu B_0}$$

$$\{\frac{i}{\lambda}\sqrt{2\mu B_0}G_9A_{11}-\left(i\sqrt{2\mu B_0}G_{10}+\frac{\rho_0}{2}\right)A_{12}-uG_7A_{1b}+G_7\phi_1\}$$

*利用的公式和结果

$$g=\frac{C_0}{2\pi}\int_0^{2\pi}\exp\left[-iat+i\lambda\cos\left(\theta+t\right)\right]dt$$

$$\frac{\partial g}{\partial\theta}=\frac{C_0}{2\pi}\int_0^{2\pi}\left[-i\lambda\sin\left(\theta+t\right)\right]\exp\left[-iat+i\lambda\cos\left(\theta+t\right)\right]dt$$

$$\frac{\partial g}{\partial\theta}\left(\theta\right)=\frac{\partial g}{\partial\theta}\left(\theta+2\pi\right)$$

$$\frac{\partial g}{\partial\lambda}=\frac{C_0}{2\pi}\int_0^{2\pi}i\cos\left(\theta+t\right)\exp\left[-iat+i\lambda\cos\left(\theta+t\right)\right]dt$$

$$\frac{\partial g}{\partial\lambda}\left(\theta\right)=\frac{\partial g}{\partial\lambda}\left(\theta+2\pi\right)$$

$$J_0\left(x\right)=\frac{1}{2\pi}\int_0^{2\pi}d\alpha\exp\left[ix\sin\alpha\right]$$

$$J_0\left(-2\lambda\sin\frac{t}{2}\right)$$

$$J_{-a}\left(-\lambda\right)J_a\left(-\lambda\right)=\frac{1}{2\pi}e^{(ia\pi)}\int_0^{2\pi}dtJ_0\left(-2\lambda\sin\frac{t}{2}\right)e^{-iat}$$

$$G_0=\frac{1}{2\pi}\int d\theta e^{-i\lambda\cos\theta}g$$

$$G_1=\frac{1}{2\pi}\int d\theta\left(\sin\theta\right)e^{-i\lambda\cos\theta}g$$

$$=-\frac{1}{i\lambda}\left[\frac{C_0}{2\pi}\left(e^{-ia2\pi}-1\right)+iaG_0\right]$$

$$G_2=\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}g$$

$$=\frac{1}{-2i}\frac{\partial}{\partial\lambda}G_0$$

$$G_3=\frac{1}{2\pi}\int d\theta\left(\sin\theta\right)e^{-i\lambda\cos\theta}\frac{\partial g}{\partial\theta}$$

$$=\frac{1}{i\lambda}\left(-\frac{1}{2}\lambda^2-\frac{\lambda^2}{4}\frac{\partial^2}{\partial\lambda^2}-\frac{\lambda}{4}\frac{\partial}{\partial\lambda}+a^2\right)G_0$$

$$=-i\left(-\frac{1}{2}\lambda-\frac{\lambda}{4}\frac{\partial^2}{\partial\lambda^2}-\frac{1}{4}\frac{\partial}{\partial\lambda}+a^2\right)G_0$$

$$G_4 = \frac{1}{2\pi}\int d\theta\left(\sin\theta\right)e^{-i\lambda\cos\theta}\,\frac{\partial g}{\partial\lambda}$$

$$= \frac{a}{\lambda}\frac{\partial}{\partial\lambda}G_0$$

$$\frac{1}{2\pi}\int d\theta\sin\theta e^{-i\lambda\cos\theta}\,\frac{\partial^2 g}{\partial\theta^2} = G_5$$

$$\frac{1}{2\pi}\int d\theta\sin\theta e^{-i\lambda\cos\theta}\,\frac{\partial^2 g}{\partial\theta\partial\lambda} = G_6$$

$$\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}\,\frac{\partial g}{\partial\theta} = G_7$$

$$\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}\,\frac{\partial g}{\partial\lambda} = G_8$$

$$\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}\,\frac{\partial^2 g}{\partial\theta^2} = G_9$$

$$\frac{1}{2\pi}\int d\theta\cos\theta e^{-i\lambda\cos\theta}\,\frac{\partial^2 g}{\partial\theta\partial\lambda} = G_{10}$$

8.4 最终结果

与 $\partial_u F$ 相关的量记为：为红色部分

$$\bar{\boldsymbol{a}} = \begin{pmatrix} a_{11} & a_{12} & a_{1b} & a_1 \\ a_{21} & a_{22} & a_{2b} & a_2 \\ a_{31} & a_{32} & a_{3b} & a_3 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2\mu B_0}ik_b\left(\frac{i}{\lambda}\sqrt{2\mu B_0}G_3 - uG_1\right) & 2\mu B_0 k_b G_4 & 0 & \sqrt{2\mu B_0}ik_b G_1 \\ \sqrt{2\mu B_0}ik_b\left(\frac{i}{\lambda}\sqrt{2\mu B_0}G_7 - uG_1\right) & 2\mu B_0 k_b G_8 & 0 & \sqrt{2\mu B_0}ik_b G_2 \\ uk_b\frac{B_0}{k_1}\left[\frac{C_0}{2\pi}\left(e^{-ia2\pi}-1\right)+iaG_0\right] & -uk_b\sqrt{2\mu B_0}\left(\frac{1}{2}\frac{\partial}{\partial\lambda}G_0\right) & -u\left(1-ik_b uG_0\right) & -uik_b G_0 \end{pmatrix}$$

与 $\partial_\mu F$ 相关的量记为：为蓝色部分

$$\bar{\boldsymbol{b}} = \begin{pmatrix} b_{11} & b_{12} & b_{1b} & b_1 \\ b_{21} & b_{22} & b_{2b} & b_2 \\ b_{31} & b_{32} & b_{3b} & b_3 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2\mu B_0}\left(\frac{i}{\lambda}\sqrt{2\mu B_0}G_5 - \frac{\rho_0}{2}\right) & -i2\mu B_0 G_6 & -u\sqrt{2\mu B_0}G_3 & \sqrt{2\mu B_0}G_3 \\ \frac{i}{\lambda}2\mu B_0 G_9 & -\sqrt{2\mu B_0}\left(i\sqrt{2\mu B_0}G_{10} + \frac{\rho_0}{2}\right) & -u\sqrt{2\mu B_0}G_7 & \sqrt{2\mu B_0}G_7 \\ -iu\sqrt{2\mu B_0}\left(-\frac{1}{2}\lambda - \frac{\lambda}{4}\frac{\partial^2}{\partial\lambda^2} - \frac{1}{4}\frac{\partial}{\partial\lambda} + a^2\right)G_0 & u\sqrt{2\mu B_0}\left(a\frac{\partial}{\partial\lambda}G_0\right) & u^2\left[\frac{C_0}{2\pi}\left(e^{-ia2\pi}-1\right)+iaG_0\right] & -u\left[\frac{C_0}{2\pi}\left(e^{-ia2\pi}-1\right)+iaG_0\right] \end{pmatrix}$$